Non Hermitian Hamiltonian with gauge-like transformation

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Abstract

The non Hermitian Hamiltonian is solved for the two quasi-exactly solvable potential by using gauge-like transformation. Possible generalization of our approach is outlined.

Keywords: Non Hermitian Hamiltonians, PT-symmetry, Khare- Mondal potential.

Resumen

El Hamiltoniano no Hermítico es resuelto para dos potenciales cuasi-exactamente resolubles mediante una transformación de norma. Se menciona una posible generalización de nuestra aproximación.

Palabras clave: Hamiltonianos no Hermíticos, PT-simetría, potencial de Khare-Mondal.

I. INTRODUCTION

The recent growth of interest in the possibility of working with non Hermitian observable in quantum theory is mainly due to the influential letters[1, 2] where the authors have observed that the spectrum of certain Hamiltonians $H = H^*$ [where * denotes both complex conjugation and transpose] seems real, discrete and bounded below. By dropping the requirement of Hermiticity but of course keeping in the invariance by the Loentz group, they open on a significantly larger class of Hamiltonians satisfying a weaker hypothesis, namely the *PT*-invariance. This more flexible condition amounts to the commutability of the Hamiltonian *H* and the composite operator *PT* whose components consist of one linear operator *P* and another anti-linear operator *T*. It has been shown that, despite the lack of Hermiticity, many *PT*-symmetric Hamiltonians still have a whole real, discrete and bounded below spectrum [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

The *PT*-symmetry is not a sufficient, or necessary condition for the reality of energy spectrum. This has been shown by several non PT-symmetric complex potential models [3, 12] with real energy spectrum. This idea has been further developed by Mostafazadeh [13]. In this noteworthy work he has introduced the concept of *η*-pseudo-Hermiticity, $\eta H \eta^{-1} = H^*$ where *η* is a Hermitian linear automorphism. It is observed that *η*-pseudo-Hermiticity is a necessary but not sufficient condition to ensure the reality of energy spectrum for a non Hermitian complex potential. The necessary and sufficient condition for the reality of energy spectrum of any Hamiltonian is that the Hamiltonian admits a complete set of bi-orthonormal eigenvectors.

Most of the papers have discussed the solution of the Hamiltonian of type $H = p^2 + V(x)$. There are few papers in which the authors have studied the case of the Hamiltonian of the type, $H = [p + \xi g(x)]^2 + V(x)$, which are very important in quantum mechanics [14, 15]. In the context of studies of delocalization phenomena, the model of Hatano and Nelson [15] has attracted a lot of interest recently [15]. It is defined in one dimension by the non-Hermitian Hamiltonian $H = [p + \xi g(x)]^2 + V(x)$, where *g* is a real parameter connected to an externally applied magnetic field, and *V(x)* is a random potential. It has been demonstrated numerically that at a certain critical value $g=g_c$ a localized wave function turns into a delocalized one, and it has been suggested that this behavior signals the occurrence of a delocalization phase transition. In this present contribution, we have considered a general model of a non-Hermitian Hamiltonians which have real spectrum and have studied the energy spectrum.

The organization of the paper is as follows. We have discussed the eigenvalue and eigenfunctions of Khare-Mondal [16] and Khare-Mondal-like [17] potential in Section II. In Section III, the gauge-like transformation on non-Hermitian Hamiltonians have been discussed. The Section IV has been kept for conclusions and discussions.

II. KHARE-MONDAL POTENTIAL

We consider the generalized potential

$$V(x) = -(\xi S^{(\sigma)}(x) - iM)^2. \qquad (1)$$

Where $S^{(\sigma)}(x) = \dfrac{e^{2x} + \sigma \times e^{-2x}}{2}$, $\sigma = +1, -1$ and M is positive integer. For $\sigma = 1$, $V(x) = -(\xi \cosh(2x) - iM)^2$ is Khare-Mondal potential and for $\sigma = -1$, $V(x) = -(\xi \sinh(2x) - iM)^2$ is Khare-Mondal-like potential. Let us remember that

$$S^{(\sigma)}\left(i\frac{\pi}{2} - x\right) = -\sigma \times S^{(\sigma)}(x),$$

and

$$S^{(\sigma)}(-x) = \sigma \times S^{(\sigma)}(x).$$

When $\sigma = 1$ (1) is invariant under the transformation $x \to i\dfrac{\pi}{2} - x$, but not PT-invariant and when $\sigma = -1$, (1) is PT-invariant under the transformation $x \to -x$ [16, 17]. The potential given in (1) is quasi-exactly solvable. We restrict our discussion up to case $M = 4$. The real energy eigenvalues and the eigenfunctions for the potential given in (1) are given by [16, 17]

When $M = 1$

$$E^{(\sigma)} = 1 - \sigma\xi^2, \tag{2}$$

$$\psi^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}S^{(\sigma)}(x)\right]. \tag{3}$$

When $M = 2$

$$E_\mu^{(\sigma)} = 3 - \sigma\xi^2 + 2\mu\xi S^{(-\sigma+i(1+\sigma))}(0), \quad \mu = \pm 1, \tag{4}$$

$$\psi_\mu^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}S^{(\sigma)}(x)\right] \times \left[S^{\frac{\mu}{2}((1+\sigma)+i(1-\sigma))}\left(-\frac{x}{2}\right)\right]. \tag{5}$$

When $M = 3$

$$E_0^{(\sigma)} = 5 - \sigma\xi^2, \tag{6}$$

$$E_\mu^{(\sigma)} = 7 - \sigma\xi^2 + 2\mu\sqrt{1 - 4\sigma\xi^2}, \tag{7}$$

$$\psi_0^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}S^{(\sigma)}(x)\right] \times S^{(-\sigma)}(x) \tag{8}$$

$$\psi_\mu^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}S^{(\sigma)}(x)\right] \times \left[2S^{(\sigma)}(x) - \frac{i}{\xi}(1 + \mu\sqrt{1 - 4\sigma\xi^2})\right]. \tag{9}$$

When $M = 4$

$$E_\mu^{(\sigma)} = 11 - \sigma\xi^2 - 2\mu\xi S^{(-\sigma+i(1+\sigma))}(0) + 4\mu\sqrt{1 - \mu\xi S^{(-\sigma+i(1+\sigma))}(0)}, \tag{10}$$

$$\psi_\mu^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}S^{(\sigma)}(x)\right] \times \left[S^{\frac{\mu}{2}((1+\sigma)+i(1-\sigma))}\left(-\frac{x}{2}\right)\right]$$
$$\times \left[S^{(\sigma)}(x) - \frac{i}{\xi}\sqrt{1 - \mu\xi S^{(-\sigma+i(1+\sigma))}(0) + \xi^2}\right]. \tag{11}$$

III. GAUGE-LIKE TRANSFORMATION

Let the Hamiltonian be of the form

$$H = [p + \xi S^{(\sigma)}(x)]^2 + V(x), \quad (2m = 1 = \hbar). \tag{12}$$

Applying the properties of commutator bracket

$$\left\{p + \xi S^{(\sigma)}(x)\right\}\exp[if(x)] - \exp[if(x)]\left\{p + \xi S^{(\sigma)}(x)\right\}$$
$$= [p + \xi S^{(\sigma)}(x), \exp[if(x)]],$$
$$= [p, \exp[if(x)]],$$
$$= \exp[if(x)]\partial_x f(x).$$

Multiplying by $\exp[-if(x)]$ on both sides we get

$$\exp[-if(x)](p + \xi S^{(\sigma)}(x))\exp[if(x)]$$
$$= p + \xi S^{(\sigma)}(x) + \partial_x f(x). \tag{13}$$

We have

$$\exp[-if(x)][p + \xi S^{(\sigma)}(x)]\exp[if(x)] = p - \xi S^{(\sigma)}(x), \tag{14}$$

where $f(x) = -2\xi \int S^{(\sigma)}(x)dx$.

Again

$$\exp[-if(x)][p + \xi S^{(\sigma)}(x)]^2 \exp[if(x)]$$
$$= \exp[-if(x)][p + \xi S^{(\sigma)}(x)]\exp[if(x)]\exp[-if(x)] \times$$
$$[p + S^{(\sigma)}(x)]\exp[if(x)],$$
$$= [p - iS^{(\sigma)}(x)][p - iS^{(\sigma)}(x)],$$
$$= [p - iS^{(\sigma)}(x)]^2.$$

Similarly,

$$\exp[-if(x)][p + \xi S^{(\sigma)}(x)]^n \exp[if(x)]$$
$$= [p - \xi S^{(\sigma)}(x)]^n,$$

for any natural number n. We shall now discuss a Hamiltonian of the type

$$H = [p + \xi S^{(\sigma)}(x)]^2 - (\xi S^{(\sigma)}(x) - iM)^2. \quad (15)$$

The eigenvalue equation is

$$H\varphi_n(x) = E_n\varphi_n(x). \quad (16)$$

Applying a suitable transformation

$$\exp\left[-i\frac{f(x)}{2}\right] H \exp\left[i\frac{f(x)}{2}\right]$$
$$= p^2 - (\xi S^{(\sigma)}(x) - iM)^2, \quad (17)$$

where

$f(x) = -\xi S^{(-\sigma)}(x)$, remembering that

$$\frac{d}{dx}\left[S^{(-\sigma)}(x)\right] = 2S^{(\sigma)}(x).$$

Now $\varphi_n(x)$ becomes

$$\varphi_n(x) = \exp\left[i\frac{\xi}{2}S^{(-\sigma)}(x)\right] \times \psi_n(x), \quad (18)$$

where $\psi_n(x)$ are eigenfunctions of the potential (1). For this transformation, the equations (2), (4), (6), (7) and (10) will be the same and from which we shall obtain the eigenvalues of the Hamiltonian (15) and equations (3), (5), (8), (9) and (11) become:

When $\underline{M = 1}$

$$\psi^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}\exp(2x)\right]. \quad (19)$$

When $\underline{M = 2}$

$$\psi_\mu^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}\exp(2x)\right] \times \left[S^{\frac{\mu}{2}((1+\sigma)+i(1-\sigma))}\left(-\frac{x}{2}\right)\right]. \quad (20)$$

When $\underline{M = 3}$

$$\psi_0^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}\exp(2x)\right] \times S^{(-\sigma)}(x), \quad (21)$$

$$\psi_\mu^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}\exp(2x)\right] \times$$
$$\left[2S^{(\sigma)}(x) - \frac{i}{\xi}(1 + \mu\sqrt{1-4\sigma\xi^2})\right]. \quad (22)$$

When $\underline{M = 4}$

$$\psi_\mu^{(\sigma)}(x) \propto \exp\left[i\frac{\xi}{2}\exp(2x)\right] \times \left[S^{\frac{\mu}{2}((1+\sigma)+i(1-\sigma))}\left(-\frac{x}{2}\right)\right] \times$$
$$\left[S^{(\sigma)}(x) - \frac{i}{\xi}\sqrt{1 - \mu\xi S^{(-\sigma+i(1+\sigma))}(0) + \xi^2}\right]. \quad (23)$$

Now for the case $M=1$ the eigenvalues of (15) are real. When $M=2$, $\sigma=-1$ the eigenvalues of (15) are real and for $m=2$, $\sigma=1$ the eigenvalues of (15) are complex. When $M=3$, $\sigma=-1$ the eigenvalues of (15) are always real and for $M=3$, $\sigma=1$ the eigenvalues of (15) are complex for $|\xi| >1/2$. Case $M=4$ being similar to that of case $M=2$.

IV. CONCLUSION

We have discussed the non-Hermitian Hamiltonian for Khare-Mondal and Khare-Mondal-like potential. We have also discussed the PT-symmetry of this Hamiltonians. Finally, we emphasize that the method discussed here can be extended to other potentials.

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