Half, Average, and most probable lives of filament lamps

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Abstract

The three lives of the tungsten filament lamps viz. the half life, the average life and the most probable life are shown analytically to coincide.

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In a previous paper \cite{1} the relevant expressions demonstrating the approximate equality of the half life, average life and most probable life for incandescent bulbs derived by Leff \cite{2} were further elaborated for the benefit of students. However, since Leff’s model was an empirical generalization of the radioactive law it does not follow from the principles of statistics and all the said derivations cannot be done analytically. The present note extends such calculations to a physically realistic tungsten evaporation model \cite{3, 4} which can be deduced from the laws of statistics and leads to analytical derivations.

In our evaporation model the thermal ejection of an atom corresponds to “failure” and nondecay of an atom corresponds to “success” for a binomial distribution. By making a normal approximation and assuming that the bulb fails when the undecayed fraction of atoms falls below a critical value we derived the following expression for the survival probability:

\begin{equation}
S (\tau) = 0.5 \left[ 1 + \text{erf} (w_c) \right], \quad w_c = 3 \left( 1 - \tau \right).
\end{equation}

Here \text{erf} stands for the error function \cite{5}, the dimensionless time interval \( \tau = \frac{t}{t_{1/2}} \) is experimental elapsed time and \( t_{1/2} \) is the half life in hours. By definition, at the half life point \( \tau = 1 \) we have

\begin{equation}
S (1) = \frac{1}{2}.
\end{equation}

Next, to find the most probable life \( \tau_m \) we report the general rates of change

\begin{equation}
R(\tau) \equiv \frac{dS}{d\tau} = \frac{3}{\sqrt{\pi}} \exp \left( -w_c^2 \right),
\end{equation}

\begin{equation}
R(\tau) \equiv \frac{dR}{d\tau} = 18(1 - \tau) R(\tau).
\end{equation}

By setting \( \frac{dR}{d\tau} \) as zero we identify the most probable life

\begin{equation}
\tau_m = 1.
\end{equation}

Finally, to determine the average life \( \tau_{av} \) we use the definition

\begin{equation}
\tau_{av} = \frac{\int_0^\infty \tau R(\tau) d\tau}{\int_0^\infty R(\tau) d\tau}.
\end{equation}

This can be evaluated by making the transformation \( \tau = (1 - w_c/3) \) in the range \( 3 \geq w_c \geq -\infty \):

\begin{equation}
\tau_{av} = \frac{1}{3} \int_{-\infty}^{\infty} \left( 1 - \frac{w_c}{3} \right) \frac{3}{\sqrt{\pi}} \exp \left( -w_c^2 \right).
\end{equation}
\[
\tau_{av} = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{9}{\sqrt{2\tau}} \right) \right] - \frac{e^{-9}}{6\sqrt{\pi}} ,
\]
\[
\tau_{av} = 1 - O\left( e^{-9} \right) = 1 - O\left( 10^{-4} \right).
\]

Hence our expressions for \( \tau_{1/2} \), \( \tau_m \) and \( \tau_{av} \) accurately coincide within a tiny correction factor of order \( 10^{-4} \). The basic reason for such remarkable equality is that the expression for \( R(\tau) \) written in Eq. (3) is symmetrical about the point \( \tau = 1 \). Thus, Leff’s remark that “light bulb labels are believable” has been further strengthened by the present note.

REFERENCES