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The Virial Theorem and its applications in the teaching of Modern Physics

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Abstract

The Virial Theorem is receiving scarce attention in the teaching of classical mechanics, and intermediate mechanics and in general physics courses. In this work we present a proposal for including this theorem in the contents of Gravitation or Kinetic Theory of Gases in general physics courses, and we illustrate the importance of the theorem with a sample set of applications in mechanics, introductory quantum mechanics, thermodynamics and astrophysics. Two relevant cases, the applications of the theorem to the presence of the so-called *dark matter* in the Universe, and to the study of the stability condition of the driven inverted pendulum are presented.

Keywords: Virial Theorem, Teaching of Classical Mechanics, Dark Matter, Inverted Pendulum.

Resumen

El Teorema del Virial está recibiendo escasa atención en la enseñanza de la física general, de la mecánica intermedia y de la mecánica clásica. Este trabajo presenta una propuesta para la enseñanza de este teorema en el estudio de la Gravitación o en la Teoría Cinética de Gases a nivel de física general universitaria, y se demuestra su importancia con aplicaciones en mecánica, mecánica cuántica, termodinámica y astrofísica. Como casos importantes se presenta la aplicación de este teorema a la existencia de la llamada *materia oscura* en el Universo y al estudio de la condición de estabilidad del péndulo invertido e impulsado por una fuerza externa.

Palabras Claves: Teorema del Virial, Enseñanza de la Mecánica Clásica, Materia Oscura, Péndulo Invertido.

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I. INTRODUCTION

The Virial Theorem of classical mechanics has been successfully applied in the last century to a number of relevant physics problems, mainly in astrophysics, cosmology, molecular physics and quantum mechanics and in statistical mechanics. In spite of its remarkable usefulness, numerous applications, simplicity, and importance, this theorem is not usually presented in physics courses for scientists and engineers, or it is considered as an optional topic [1] or not even mentioned at all in a good number of mechanics textbooks [2, 3, 4, 5]. A quick survey recently conducted among 12 physics lecturers showed that this theorem is rarely mentioned, and although considered very relevant by a few of them, almost all held the opinion that treating other topics of classical mechanics leave no room for the theorem. Interestingly enough the Virial Theorem was already included, as an advanced topic, in the celebrated Berkeley University Course textbook of Mechanics [6] written by C. Kittel *et al*. and published back in 1964, and written for first-year physics undergraduates. Indeed, the proof and applications of the theorem lie well within the realm of knowledge accessible to physics students taking courses of Modern Physics, usually after a formal course of newtonian mechanics, or just in parallel with an intermediate course of classical mechanics at the undergraduate second year, or just after such course. In the present work we propose the explicit inclusion of the Virial Theorem in the contents of Introductory Mechanics and Modern Physics. Two opportunities for applying the theorem are, for instance, the hypothesis of the presence of *dark matter* in the Universe, and the very important problem of finding the temperature of the interior of the Sun. From our experience, we think that both lecturers and students have to do a serious effort to provide accessible explanations for these cosmological and astrophysical subjects. On the contrary, what the students are able to grasp from the usual rhetoric presentation of the dark matter hypothesis -without resorting to the Virial Theorem- is far from being acceptable, that is the subject becomes truly *dark* for the students.

This paper is organized as follow. We first introduce the Virial Theorem, and then present its classical proof and a proposal for its teaching in physics basic courses. After that we present some applications of the theorem in quantum mechanics and thermodynamics. Finally we show the application of the theorem to the presence of dark matter and to the study of the stability condition for an inverted pendulum.

II. THE THEOREM

The Virial Theorem of R. E. Clausius [7] owes its name to the word "vires", the Latin for force, and in fact it is an easy theorem to interpret; its mathematical proof being too rather easy to follow. In spite of not being a mechanics principle the theorem and its applications ranges from the quantum world to the largest and more massive objects of the Universe, and have attracted considerable attention in the last 50 years. The theorem refers to a system of interacting particles whose well-defined time-average kinetic energy $<T>$ we are going to obtain. Both the position $\boldsymbol{r}_i$ and the velocities $\boldsymbol{v}_i$ of the interacting particles are assumed to be bound over long time (for instance, you are not expected to apply the theorem to a batted base-ball "flying" away from the stadium and escaping from Earth). With such bounding assumptions we are certain to get the classical time-average value for the potential energy $<V>$ of a system of particles if we are able to assess its classical mean kinetic energy $<T>$.

Clausius assigned the name *virial* to the quantity denoted C and defined by:

$$C = \sum_i \boldsymbol{P_i} \cdot \boldsymbol{r_i} \,, \tag{1}$$

i.e., the sum of the scalar product of each particle momentum $\boldsymbol{P_i}$(t) by its position $\boldsymbol{r}_i$(t).

In its most frequently quoted form, the theorem of Clausius simply states that for the system of interacting bound particles we can always write:

$$2\langle T\rangle = -\langle V\rangle \,. \tag{2}$$

The latter expression in fact corresponds to a potential proportional to $1/r$; which then implies that the theorem can be applied to very important gravitationally bound systems, such as the system of stars in a *spiral galaxy*, to a *cluster of galaxies* or even to a *globular cluster*. This theorem holds in a more general case for a system of particles interacting with a potential of the form $V \propto 1/r^{\alpha}$, in which case its expression becomes:

$$2\langle T\rangle = -\alpha\langle V\rangle \,. \tag{3}$$

A. Proposal of teaching in General Physics

Many texts of introductory classic mechanics do not usually make reference at all to the Virial Theorem, or simply do it by introducing a scalar function without relating it to some previously known physical phenomenon. But we know that for truly significant learning it is important that the teaching process keeps some relation with knowledge already acquired by the student [8]. Consequent with this assertion, our pedagogical proposal is to associate the scalar function from the Virial Theorem to conceptual and mathematical physics of bodies moving under central forces, like in Gravitation, given that this scalar function is related to the radial translation of bodies.

If we consider a particle of mass m and velocity $\boldsymbol{v}$ that moves under the action of a central force $\boldsymbol{F}$ of the type:

$$\boldsymbol{F} = -(\gamma Mm/r^2)\ \hat{\boldsymbol{r}} \,, \tag{4}$$

where $M >> m$ is the mass of the object that generates the interaction. The lineal momentum of the mass m is:

$$\boldsymbol{P} = m\boldsymbol{v}, \quad \boldsymbol{v} = \upsilon_\theta\,\hat{\boldsymbol{\theta}} + v_r\,\hat{\boldsymbol{r}} \,, \tag{5}$$

where $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{r}}$ are unit vectors in polar coordinates. We can now define two quantities: a vector quantity $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{P}$ the well-known angular momentum of Newtonian mechanics, and a scalar quantity $A=\boldsymbol{P}\cdot\boldsymbol{r}$ .We can also take the temporal derivative of both quantities to study the evolution of the system. For the first quantity, $\boldsymbol{L}$, it is known that for a central force parallel to $\boldsymbol{r}$, $\dot{\boldsymbol{L}} = 0$, that is, the angular momentum is conserved. On the other hand, taking the first derivative of A, we obtain:

$$\dot{A} = m\,\dot{\boldsymbol{v}}\cdot\boldsymbol{r} + m\,\boldsymbol{v}\cdot\dot{\boldsymbol{r}} \,, \tag{6a}$$

$$= m\,\dot{\boldsymbol{v}}\cdot\boldsymbol{r} + mv^2, \tag{6b}$$

$$= \boldsymbol{F}\cdot\boldsymbol{r} + mv^2, \tag{6c}$$

where $mv^2=2T$, T being the kinetic energy of the particle.

If the force is conservative then $\boldsymbol{F} = -\frac{\partial V}{\partial r}\,\hat{\boldsymbol{r}}$,V being the gravitational potential energy of the system given by $V= -\frac{\gamma mM}{r}$.

Replacing Eq. (4) into (6c) we get:

$$\dot{A} = -V + 2T \,. \tag{7}$$

As the angular momentum is conserved, then $L = mrv$ is a constant, and r and v are values bound in time. Taking the temporary average of (7):

$$\langle \dot{A}\rangle = \frac{1}{\tau}\int_0^{\tau}\frac{dA}{dt}dt = \frac{A-A_0}{\tau} \,,$$

we get that $\langle \dot{A}\rangle \to 0$ for $\tau >> (A-A_0)$. This implies that:

$$\langle \dot{A} \rangle = -\langle V \rangle + 2\langle T \rangle = 0 \Rightarrow \langle V \rangle = -2\langle T \rangle . \qquad (8)$$

This equation is the relationship between the average kinetic energy and the average potential energy or *virial* (the force components are given by the derivatives of the potential energy V) of the particle under the action of a central force (and α=1 in Eq. 3).

In this way we can study the behaviour of a particle of mass m that moves under the action of a central force, considering two aspects: the *rotational* movement related to the angular momentum, and the *traslational* movement related to the virial. In the first case there is a constant of motion: $\dot{\boldsymbol{L}} = \boldsymbol{0}$. A result representing the law of conservation of L**.** In the second case, although $\langle \dot{A} \rangle = 0$ is *not* a conservation theorem, it could be considered like an alternative way to the study of the adiabatic invariance [9] that will lead us to establish a statistical relationship between the kinetic energy average and the potential energy average of the particle.

III. APPLICATIONS

In this section we present a set of applications of the Virial Theorem. We would like to emphasize that the set of applications presented here is far from being exhaustive and the reader can find a lot more in physics journals [10, 11, 12, 13, 14, 15]. Starting with the important case of the determination of the temperature of the interior of a star, that is in fact the first application in the *advanced topic* of the Mechanics Physics Course of Berkeley University [6]. We will also consider below the very important case of the cosmological hypothesis known as the *Dark Matter* and a few other applications in quantum mechanics and mechanics.

A. Temperature of the interior of a star

Finding the temperature at the surface of the Sun is a standard example presented in all Modern Physics courses as an application of Planck Quantum Theory of Radiation. Less known is the calculation of the temperature of the interior of a star, a case that is best and most effectively treated using the Virial Theorem [6]. Assuming that a star is a sphere of radius R, and mass $M_{s,}$ its total gravitational potential energy V is found using a well-known relation of general physics courses.

$$V = -\frac{3GM_s^2}{5R} . \qquad (9)$$

With the safe assumption that a single atom moving in the interior of the star has a mean kinetic energy $\langle K_e \rangle$ given by energy equipartition by

$$\langle K_e \rangle = \frac{3}{2} k \langle T_S \rangle , \qquad (10)$$

where $\langle Ts \rangle$ is the mean temperature over the interior of the star, and k is *Boltzmann Constant*.

If N is the total number of atoms in the star then the application of the *Virial Theorem* gives

$$-\langle V \rangle / 2 \approx -\frac{3GM_s^2}{10R} = N \frac{3k\langle T_S \rangle}{2} , \qquad (11)$$

therefore

$$\langle T_S \rangle \approx \frac{GM_s^2}{5kNR} = \frac{GM_s m}{5kR} , \qquad (12)$$

where $m = M_s/N$ is the average mass of an atom of the star. Typical stars such as our Sun contain mostly hydrogen atoms (~61 %) and helium atoms (38 %), and we may therefore approximate the atom mass m = 2.2x10 $^{-27}$ [Kg]. The mass of the Sun is about $M_{\odot}$=2x10^{30} [Kg] and its radius may be taken as 70 million kilometers. Introducing these constants in equation (12) we get an estimate of our Sun interior temperature (10^{7} [K]) which coincides with estimates using other physics phenomena that take place in the star (e.g. nucleo synthesis). As Kittel *et al.* comment [6] this is a remarkable result given the simple calculation required, and the small amount of experimental data demanded, all of which is readily available from measurements in our own planet: not necessary to get close to the Sun!

B. Quantum Mechanics

The Virial Theorem is frequently applied in problems of quantum mechanics in textbooks which are widely used. It is also of standard use in molecular physics. As an illustration we consider below the rather simple case of finding the average size of the radial *eigenfunctions* of the hydrogen atom [16, 17, 18]. Other not so simple, and more useful, cases of applications can be found in references [10, 11, 17].

For any eigenfunction φ – of eigenenergy E – of the hydrogen atom the *quantum expectation value* $\langle \varphi | H | \varphi \rangle$ of the Hamiltonian is equal to the sum of the *quantum expectation values* $\langle T \rangle_{\varphi}$, $\langle V \rangle_{\varphi}$:

$$\langle H \rangle_{\varphi} = E = \langle T \rangle_{\varphi} + \langle V \rangle_{\varphi} ,$$
$$= \langle P^2/2m \rangle - \langle e^2/r \rangle . \qquad (13)$$

Now the *quantized version* of the Virial Theorem can be shown to be [10, 17]:

$$\langle T \rangle_{\varphi} = -1/2 \langle V \rangle_{\varphi} , \qquad (14)$$

where the reader should not take *quantum expectations values* as *classical averages*. Replacing above we get:

$$E = < V >/2 = - < e^2/r >. \quad (15)$$

Moreover, if the atom is in the n^{th} eigenstate its energy is well-known [17] to be

$$E_n = -2m\pi^2 e^4/(n^2 h^2) = E. \quad (16)$$

The last two equations finally give the sought result for the average radius of the atom

$$<1/r>_n = 1/(a_o n^2), \quad (17)$$

where a_0 is the radius of the atom of Bohr's model.

Note that, although $<1/r>$ differ from $1/<r>$ the two averages are of the same order of magnitude and thus we finally get:

$$< r_n > \approx n^2 a_o \quad , \quad (18)$$

this is the radial distance where the probability of finding the electron in the hydrogen atom reaches its maximum value.

Other very important cases of application of Virial Theorem in quantum mechanics are the proof of the origin of the *chemical bond* in a molecule and the analysis of canonical examples of quantum mechanics such as the Harmonic Oscillator [17].

C. Kinetic theory of gases

If we consider a gas confined into a recipient, the interactions between molecules of the gas will be bound by the walls of the recipient. Let us evaluate the terms in the *r.h.s.* of Eq (6c).

Taking a force differential on the gas molecules, defined by the pressure P exerted by the wall of the recipient in a differential area dA we may write

$$d\boldsymbol{F} = P dA\,\hat{\boldsymbol{n}}, \quad (19)$$

so the total force will be:

$$\boldsymbol{F} = \int P dA\ \hat{\boldsymbol{n}}. \quad (20)$$

The term $\boldsymbol{F}\cdot\boldsymbol{r}$, Eq (6c) is, together with (20):

$$\boldsymbol{F}\cdot\boldsymbol{r} = P\int \boldsymbol{r}\cdot dA\,\hat{\boldsymbol{n}} \ , \quad (21)$$

and applying the well-known Gauss theorem of vector calculus we get**:**

$$\int \boldsymbol{r}\cdot dA\,\hat{\boldsymbol{n}} = \int (\nabla\cdot\boldsymbol{r})dv = 3V \quad (22)$$

where V is the volume span by the gas.

The remaining term of (6c), that is mv^2, is twice the value of kinetic energy. Again from the theorem of energy equipartition the average kinetic energy of an ideal gas is given by Eq. (10). If we now take $\langle \dot{A} \rangle \to 0$ in (6c), we get $\langle F\cdot r\rangle + \langle mv^2 \rangle = 0$. Replacing from Eqs. (21), (22), (10) into this equation and eliminating the common factor 3/2, we arrive to the well-known equation of the ideal gases:

$$\langle P\rangle V = NkT, \quad (23)$$

where N is the number of mol and $\langle P\rangle$ the macroscopic pressure average.

IV. THE DARK MATTER HYPOTHESIS

A cluster of galaxies is a huge physical system consisting of galaxies that are gravitationally bound. Thousands of galaxy clusters are known to exist and have been catalogued since 1950, their typical size and solar mass being 1-5 Mpc and 2-9 $\times 10^{14} M_\odot$ respectively. The spiral galaxies themselves are bound systems too, gravitationally stable and formed by stars and interstellar gas. The main portion of luminous and observable matter in a galaxy is gathered in a thin disk where stars and gas rotate, in almost circular orbits, about the galactic centre.

Let M be the total mass of the galaxy (assumed to be concentrated at the centre), and let v and R the speed and radius of the galaxy, respectively.

Consider a star of mass m orbiting at the periphery of the galaxy, under the gravitational attraction of the galactic core and in dynamic equilibrium. The gravitational force on it is of the same magnitude as its centripetal force [19]:

$$\frac{mv^2}{R} = \frac{GMm}{R^2} \ \text{or}\ M = \frac{Rv^2}{G}, \quad (24)$$

here M is in fact the total mass inside the orbit of the star that produces the interaction. Therefore:

$$v = \sqrt{\frac{GM}{R}}. \quad (25)$$

Since the evolution of the Universe is discussed in cosmology in terms of mass density, it is useful to relate the velocity of the star with the mass density $\rho(R)$, for a given mass distribution $M(R)$, in Eq. [24]:

$$v = \sqrt{\frac{GM(R)}{R}} \propto \sqrt{G} = const. \quad (26)$$

On the other hand the radial density distribution is known to be

$$\rho(R) \propto R^{-2}. \tag{27}$$

If we apply the Virial Theorem to stars, galaxy clusters or any other gravitationally bound object, we may write:

$$\langle T \rangle = -\frac{1}{2}\langle V \rangle. \tag{28}$$

In order to obtain a relationship between the star speed and its mass, we consider the galaxy as a spherical distribution of total mass *M*, radius *R* and average density ρ. Then the potential energy assumes the well-know form [15],

$$V = -\frac{3}{5}\frac{GM^2}{R}. \tag{29}$$

For *N* stars of mass *m* in the galaxy, the total kinetic energy is:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2. \tag{30}$$

Applying the Virial Theorem to each star we obtain:

$$-\frac{m}{N}\sum_{i=1}^{N} v_i^2 = \frac{V}{N}, \tag{31}$$

where the speed includes the radial, zenithal and azimuthal speed components [20], therefore we may write for the speed average:

$$\langle v^2 \rangle = \langle v_r^2 \rangle + \langle v_\theta^2 \rangle + \langle v_\varphi^2 \rangle = 3\langle v_r^2 \rangle, \tag{32}$$

and since we are dealing with statistical averages, we may then write:

$$\frac{1}{N}\sum_{i=1}^{N} v_i^2 = \langle v_i^2 \rangle = 3\langle v_r^2 \rangle = 3\sigma, \tag{33}$$

where σ is the standard deviation. Replacing now from Eqs. (29) and (30) into the expression (28) for the virial:

$$-3m\sigma_r^2 \approx -\frac{3}{5}\frac{GM^2}{NR}, \tag{34}$$

and taking $M=N\,m$ we finally get an expression for the speed dispersion:

$$\sigma_r^2 \approx \frac{GM_{virial}}{5R}. \tag{35}$$

This equation is in fact a relationship between the galaxy speed dispersion and its mass. Now using the Doppler shift of the spectral lines of the radiation of the interstellar gas in the galaxy, it is possible to accurately assess the rotation speed of different regions of the galaxy. Note that such spectrum could be plotted on top of the continuous spectrum given by the distribution of temperatures in the galaxy. If one compares the results obtained for the galaxy dispersion speed using the electro-magnetic spectrum analysis with the one obtained using the Virial Theorem (34), one gets that for the same mass value *M* the speed given by the first procedure is much larger than the one given by the second one.

F. Zwicky [21] was the pioneer of applying the Virial Theorem to cluster of galaxies using a similar analysis to the preceding one. He found that there was about 400 times more mass than the expected one given by the spectrum analysis. He correctly concluded that there must be some extra non-luminous matter in the cluster, that extra mass being responsible for the larger observed speeds.

Note that the mass of a galaxy can also be estimated from its luminosity and its surface temperature [20]. Knowing the galaxy size *R*, it is possible to determine the gravitational acceleration $g = \frac{GM}{R^2}$ at the galaxy surface. From this, the mean density of the galaxy can be found and be compared with the critical density of the Universe given by

$$\rho_c \equiv \frac{3H_0^2}{8\pi G}, \tag{36}$$

where H_o is Hubble Constant, an equation derived from Friedmann Equations [20].

Again, the density obtained using the Virial Theorem is larger than the critical density and therefore we need the *Dark Matter Hypothesis* or any other hypothesis to account for an explanation of such difference.

V. A SPECIAL CASE: THE INVERTED PENDULUM

A few years ago Mata, *et al*. [22], found the average of the fast time dependent Hamiltonian of a driven inverted pendulum of length *l* with an oscillatory pivot, point mass *m*, as shown in Figure 1.

From the geometry shown in Figure 1 we may write

$$\mathbf{r} = \mathbf{R} + \mathbf{l}, \tag{37}$$

$$\mathbf{R} = z(t)\,\hat{\mathbf{k}}, \tag{38}$$

$$\mathbf{l} = l\sin\theta\,\hat{\mathbf{j}} + l\cos\theta\,\hat{\mathbf{k}}. \tag{39}$$

The pendulum moves in the y-z plane, its velocity **v** being:

$$\boldsymbol{v} = l\,\dot{\theta}\cos\theta\,\hat{\boldsymbol{j}} + (\dot{z} - l\dot{\theta}\sin\theta)\,\hat{\boldsymbol{k}}\,. \qquad (40)$$

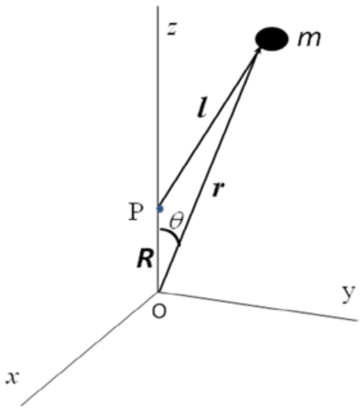

FIGURE 1. An inverted pendulum of length *l* with oscillating pivot P of vector ***R***.

The expressions for the kinetic and potential energy are, respectively:

$$T = \frac{m}{2}[(l\dot{\theta} - \dot{z}\sin\theta)^2 + \dot{z}^2\cos^2\theta], \qquad (41)$$

$$V = mg\,(z + l\cos\theta)\,, \qquad (42)$$

while their averages are:

$$\langle T\rangle = \frac{m}{2}[l^2\langle\dot{\theta}^2\rangle - 2l\langle\dot{\theta}\dot{z}\sin\theta\rangle + \langle\dot{z}^2\rangle], \qquad (43)$$

$$\langle V\rangle = mg(\langle z\rangle + l\cos\langle\theta\rangle). \qquad (44)$$

For small oscillations, the potential $V \propto 1/r^{\alpha}$ is of the form $V \propto 1/r^2$. If we set $\alpha = -2$ in the virial Eq. (3) and use the the small angle approximations: $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$, for small pendulum oscillations in (43) and (44) we obtain:

$$\langle\dot{z}^2\rangle = 2gl\left(1 - \frac{\langle\theta^2\rangle}{2}\right) - l^2\langle\dot{\theta}^2\rangle\,. \qquad (45)$$

For the position $\theta = 0$ the gravitational potential is a maximum, and the *stable* point condition, represented by $\langle\dot{z}^2\rangle$, in Eq. (45), becomes:

$$\langle\dot{z}^2\rangle = 2\,gl. \qquad (46)$$

This result is consistent with the one obtained by Mata *et al*. when the pivot motion is sinusoidal. These authors found a stable point stable when:

$$\langle\dot{z}^2\rangle > gl. \qquad (47)$$

It is clear that a good result can be obtained without using the Hamiltonian or the Lagrangian or the system. One can use instead simple concepts introduced in general physics plus the Virial Theorem (from the standpoint of the theorem: the maximum average potential that reaches the system is equal to the average kinetic energy).

VI. CONCLUSIONS

In this work we have emphasized the importance of the Virial Theorem and its practical applications in classical mechanics, quantum mechanics and thermodynamics. This theorem is an important theoretical referent for the study of open problems in modern physics, as in the case of the presence of *dark matter* in the Universe. For such reasons, we included in this work, as a pedagogical proposal, the application of the Virial Theorem in systems moving under central forces, and we also showed the specific case of the inverted pendulum where we can study its behaviour with basic concepts like energy, velocity, position and with the Virial Theorem, opening the possibility that this theorem can be taught in general physics courses. We must emphasize that the teaching proposal presented in this work has not been tested with students. This could be the subject of future research work on the advantages and limitations that may arise in the implementation of this proposal.

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