Hydrostatic equilibrium in modeling the neutral atmosphere

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Abstract
This report focuses on demonstrating the approach of hydrostatic equilibrium in modeling the neutral atmosphere and ionization intensity. We showed how the altitude variations of the neutral density can be described; altitude and density at maximum ion production and optical depth of the atmosphere can be modeled using hydrostatic equilibrium. Moreover MSIS-E-90 atmospheric model results are employed for comparison purposes to see the trends of outputs of the hydrostatic approximations. As a result, we came to conclude that hydrostatic approximations could be useful at least to explain the theoretical science behind the Earth’s atmosphere.

Keywords: Hydrostatic equilibrium, momentum equation, neutral atmosphere.

I. INTRODUCTION
It is clearly understood that ionization in Earth’s atmosphere can be taking place depending on the strength of the solar radiation as well as the neutral density of the atmosphere. This is to say that ionization can vary accordingly with the variation of the intensity of the solar radiation and the density of the neutral particles (gases) of the atmosphere. Nowadays, characterizing the neutral atmosphere, which has a direct impact on the ionospheric phenomena of our planet, becomes the concern of many scholars.

To model the density and other parameters in the atmosphere, number of investigations have been conducted and came out with smart atmospheric models such as MSIS-E-90. However, theoretically, the science of neutral atmosphere is being conducted based on hydrostatic approximations.

In this report, we tried to demonstrate the gap between MSIS-E-90 model and the hydrostatic approximation by taking density of atmosphere of Bahir Dar (11°N lat, and 37°E long), Ethiopia, at a reference height of 150km, on Oct. 15, 2009 at 14UT. In doing so, we computed the hydrostatic approximations of the neutral atmospheric density \(n\), altitude of maximum ionization \(z_m\), density at maximum ionization \(n_m\) and the optical depth \(\tau\). Particularly, the comparison between MSIS-E-90 model and hydrostatic approximation results of atmospheric density and optical depth are carried out. The general picture of our approach to carry this atmospheric model is demonstrated by figure 1. Moreover the cross-section of the neutral particles used in this hydrostatic modeling is displayed by table I.

Including the introduction, this paper has four major sections. The second section gives brief descriptions on MSIS-90 atmospheric model. The third presents atmospheric theories related to hydrostatic approximations. And then, possible results and conclusions are dealt under the fourth section of our paper.

II. MSISE-90 MODEL DESCRIPTION
The MSISE model describes the neutral temperature and densities in Earth's atmosphere from ground to thermospheric heights. Below 72.5 km, the model is based...
on the MAP Handbook tabulation of zonal average temperature and pressure [1]. Below 20 km these data were supplemented with averages from the National Meteorological Center (NMC). Besides this, measurements of pitot tube, falling sphere, and grenade sounder rocket during 1947 to 1972 were treated.

Above 72.5 km, MSISE-90 is essentially a revised MSIS-86 model taking into account data derived from space shuttle flights and newer incoherent scatter results [2]. The authors of MSISE-90 model recommend the MSIS-86 model for those who are interested only in the thermosphere (above 120 km).

III. HYDROSTATIC APPROXIMATIONS

The momentum equation for a gas of density \( \rho \) and velocity \( v \) can be defined by

\[
\rho \left[ (v, \nabla) + \frac{d}{dt} \right] = f ,
\]

where \( f \) stands for total force density [5]. Basically, we consider gravitational and pressure gradient force densities that can be given by

\[
f = \rho g - \nabla p .
\]

For static atmosphere, the left hand side of the momentum equation, Eq. (1), becomes zero and we left only with a balance between gravitational force density and pressure gradient. This approximation is known as hydrostatic equilibrium [4] and it leads to have a hydrostatic equation given by

\[
\rho g = \nabla p .
\]

The above equation can be written in number density by introducing \( \rho = mn \) as

\[
mng = \nabla p ,
\]

where \( n \) is the number density and \( m \) is the molecular mass of the neutral species of the atmosphere.

A. Number density of the neutral atmosphere

Let us now try to formulate the number density of the atmosphere based on the hydrostatic approximation (equation). For spherical symmetry, the pressure gradient can be given by

\[
\nabla p = \frac{dp}{dr} e_r.
\]

So Eq. (4) can be rewritten as

\[
mng = \frac{dp}{dr} e_r .
\]

FIGURE 1. Flowchart used for modeling the neutral atmosphere.

It is known that the Earth (any planet) attracts its atmosphere by a gravitational acceleration of

\[
g = \frac{GM}{r^2} e_r = \frac{GM_r}{R^2} e_r = \frac{R^2}{r^2} g_o e_r ,
\]

where \( g_o \) is the gravitational acceleration at the surface of the Earth and \( R \) is the radius of the Earth. Substituting Eq. (7) into Eq. (6) gives

\[
- mng_o \frac{R^2}{r^2} e_r = \frac{dp}{dr} e_r ,
\]

This can be written as

\[
\frac{dp}{dr} = - mng_o \frac{R^2}{r^2} .
\]

From ideal gas law, we have \( n = \frac{p}{k_BT} \).

Using this relation and integrating with respect to \( r \) gives the altitude dependence of the atmospheric pressure and temperature as well. That is

\[
p = p_o \left[ - \frac{r}{R} mng_o \frac{R^2}{k_BT(r)} \frac{dr}{r^2} \right] .
\]

If we consider the isothermal atmosphere (\( T=\text{constant} \)) and use \( p = nk_BT \rightarrow p_o = n_o k_BT \), then we can have a
Hydrostatic approximation based model for the number density of the atmosphere. Thus we can have

\[ n = n_0 \exp \left( -\frac{z - z_o}{H} \right), \]  

where \( H = \frac{k_B T}{mg} \) is the scale height of the neutral atmosphere \( z = r - R \) and \( z_o \) is the reference altitude (height).

**Figure 2.** MSIS-E-90 model approximations of number densities of N\(_2\), O\(_2\) and O species.

**Figure 3.** Hydrostatic approximations of number densities of N\(_2\), O\(_2\) and O species

**B. Altitude and density at maximum ionization**

To model the altitude and density of the neutral atmosphere at maximum ion production, let us start from the definition of production rate. That is

\[ q = q_{m,0} \cos \theta \exp \left( t - h' - e^{-h'} \right), \]  

where \( h' = h - \ln \sec \theta \), \( \theta \) is the zenith angle and, \( q_{m,0} \) is the production rate at \( \theta = 0 \). Maximum ion production can be attained at an altitude of \( h_m \) where \( \frac{dq}{dh'} = 0 \). Hence we have

\[ q = q_{m,0} \cos \theta \exp \left( t - h' - e^{-h'} \right)(-1 + e^{-h'}) = 0. \]  

This leads to

\[ \begin{align*}
-1 + e^{-h'} &= 0, \\
e^{-h'} &= 0, \\
-h' &= \ln 1, \\
h' &= 0.
\end{align*} \]  

Since \( h' = h - \ln \sec \theta \), then we have

\[ \begin{align*}
h_m - \ln \sec \theta &= 0, \\
\Rightarrow h_m &= \ln \sec \theta.
\end{align*} \]  

Furthermore, since \( h = \frac{z_m - z_o}{H} \), then

\[ \begin{align*}
\frac{z_m - z_o}{H} &= \ln \sec \theta, \\
\Rightarrow z_m &= z_o + H \ln \sec \theta.
\end{align*} \]  

Eq. (18) refers to altitude at which maximum ion production can be taking place.

**Figure 4.** Number density of N\(_2\) molecule.

**Hydrostatic equilibrium in modeling neutral atmosphere**

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Eq. (18) refers to altitude at which maximum ion production can be taking place.
At this altitude, the neutral atmosphere has a particular density \( n_m \) that can be computed from the fact that the optical depth at maximum production becomes one [3]. That is

\[
\tau = \sigma \sec \theta H n(z), \tag{19}
\]

\[
l = \sigma \sec \theta H n_m. \tag{20}
\]

From this relation one can easily solve for \( n_m \) as

\[
n_m = \frac{1}{\sigma \sec \theta H}. \tag{21}
\]

This results

\[
\ln \frac{n_m}{n_o} = \frac{z_m - z_o}{H} \Rightarrow
\]

\[
z_m = z_o + H \ln (H \sigma n_o \sec \theta). \tag{23}
\]

Optical depth in the Earth’s atmosphere relates the intensity of solar radiation at a certain altitude \( z \) to that of the intensity at infinity [6]. It depends on both altitude and wavelength (absorption cross-section of the atmosphere).

If we have \( j \) neutral species in the atmosphere then the optical depth at an altitude \( z \) and wavelength \( \lambda \) can be defined as

\[
\tau(\lambda, z) = \sec \theta \sum_j \sigma_j(\lambda) n_j(z) H_j. \tag{24}
\]

C. Optical depth

For instance if the atmosphere is basically consists of \( \text{N}_2 \), \( \text{O}_2 \) and \( \text{O} \) species then the optical depth at 100km, \( \theta = 30^\circ \) and \( \lambda = 50\text{nm} \) can be computed as

\[
\tau(50, 100) = \sec 30^\circ \left[ \sigma_{\text{N}_2}(50) n_{\text{N}_2}(100) H_{\text{N}_2} + \sigma_{\text{O}_2}(50) n_{\text{O}_2}(100) H_{\text{O}_2} \right]. \tag{25}
\]

The ratio of the intensity at an altitude \( z \) to intensity at \( \infty \) can be computed from optical depth, \( \tau \), as

\[
\frac{I(z)}{I(\infty)} = \exp(-\tau). \tag{26}
\]

This fraction can tell us the amount of radiation absorbed in the ionization process at a specific altitude compared with intensity at the source, \( I(\infty) \).

IV. RESULTS AND DISCUSSIONS

In this report, we demonstrated how neutral atmosphere parameters such as number density could be modeled using hydrostatic approximations. We considered a specific atmosphere and simulated temperature and number density at one particular reference altitude for \( \text{N}_2 \), \( \text{O}_2 \) and \( \text{O} \) neutral species using MSIS-E-90 atmospheric model. After that, we modeled the altitude variation of the number density of the atmosphere using hydrostatic
conditions. Consequently, we computed the altitude and the density of the neutral atmosphere at which maximum ion production can be attained. Moreover, we showed the amount of radiation intensity that can be received at a certain altitude by computing optical depth of the atmosphere. In all above cases, we implemented the hydrostatic approximations by considering the density of the atmosphere at reference height 150km, Bahir Dar (11°N lat, and 37°E long), Ethiopia, Oct. 15, 2009 at 14UT.

**FIGURE 7.** Hydrostatic approximations of optical depth of the atmosphere.

**TABLE I.** Cross-sections of neutral particles used in modeling.

<table>
<thead>
<tr>
<th>λ (nm)</th>
<th>$\sigma_{N_2} \times 10^{-22} (m^2)$</th>
<th>$\sigma_{O_2} \times 10^{-22} (m^2)$</th>
<th>$\sigma_{O} \times 10^{-22} (m^2)$</th>
</tr>
</thead>
<tbody>
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<td>10</td>
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<td>2.9</td>
<td>2.3</td>
</tr>
<tr>
<td>20</td>
<td>6.4</td>
<td>8.7</td>
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</tr>
<tr>
<td>30</td>
<td>11.6</td>
<td>16.0</td>
<td>9.7</td>
</tr>
<tr>
<td>40</td>
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<td>18.7</td>
<td>10.9</td>
</tr>
<tr>
<td>50</td>
<td>23.3</td>
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</tr>
<tr>
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</tr>
<tr>
<td>100</td>
<td>0.001</td>
<td>2.0</td>
<td>0.001</td>
</tr>
</tbody>
</table>

The results of our hydrostatic modeling are illustrated by figures 3, 4, 5, 6 and 7. Accordingly, figure 3 shows the altitude variation of the number density of the atmosphere modeled by hydrostatic approximations. The comparison of this result with MSIS-E-90 model density outputs is illustrated by figures 4 for $N_2$ and 5 for O and $O_2$ species of the atmosphere. As these results show, there is some agreement between hydrostatic and MSIS-E-90 model results. This confirms the fact that hydrostatic approximations can still be considered as a means to model Earth’s atmosphere.

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**REFERENCES**