Path integrals and wave packet evolution of the Modified Caldirola-Kanai Oscillators

Akpan N. Ikot[1], Edet J. Uwah,[2] Louis E. Akpabio[1] and Ita O. Akpan[2]
[1]*Department of Physics University of Uyo, Uyo Nigeria.*
[2]*Department of Physics University of Calabar, Calabar Nigeria.*

E-mail: ndemikot2005@yahoo.com

(Received 7 February 2010; accepted 12 May 2010)

Abstract

The Lagrangian, propagator and wave function for the damped harmonic oscillator with time-dependent frequency in the frame of Caldirola-Kanai Oscillator is evaluated. We also evaluate the uncertainty relation and the evolution of the Gaussian wave packet arising from the wave function.

Keywords: Path Integrals, Caldirola- Kanai Oscillator and Wave Function.

Resumen

El propagador Lagrangiano y la función de onda para el oscilador armónico amortiguado con una frecuencia que dependiente del tiempo en el marco de Caldirola-Kanai oscilador se evalúan. También evaluamos la relación de incertidumbre y de la evolución del paquete de ondas gaussiano derivadas de la función de onda.

Palabras clave: Integrales Ruta de acceso, Caldirola-Kanai oscilador y la función de onda.

PACS: 03.65Ge, 03.63-w, 42.52-p **ISSN 1870-9095**

I. INTRODUCTION

The path integral formalism of quantum mechanics provides a systematic way of solving quantum mechanical problems and always avoids the operation methods of Schrödinger and Heisenberg. The path Integral is therefore more fundamental, more intuitive and even more flexible than the operator formalism. The basic and central concept in Feynman approach is the propagator (Green's function) of the Schrödinger equation and contains all the information about the system under investigation [1]. The path integral evaluation of some simple quantum harmonic Oscillators have attracted a considerably attention [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. However, it was Feynman [2] and Dirac [11] and other authors [3,4,10] who realizes that the integral Kernel (propagator) of the time-evolution operator can be expressed as a sum over all possible paths connecting the points q_1 and q_2 with weight factor $\exp[(i/\hbar\, S(q_1, q_2;T)]$, where S is the action.

The dissipative system is usually ascribed as having a microscopic nature [12, 13, 14, 15, 16, 17]. The study of the dissipative quantum systems as a damped harmonic oscillator was first adopted by Kanai and Caldirola [13, 14]. Several attempts have been made in understanding the dissipation system at a more fundamental level [17]. One of the simplest models of dissipation is the damped quantum harmonic oscillation with one or two degree of freedom in the frame work of Caldirola Kanai Oscillator [1, 17, 18] and its modified form [19].

Our primary objective of this paper will be to construct the Lagrangian of the modified Oscillator and use the result to derive the path integral for the damped system. With the dynamical invariant method introduced by Lewis-Risenfeld [20-22], we derive the exact wave function for the one dimensional Caldirola-Kanai Hamiltonian and then study the wave packet evolution arising from these propagators [23].

The organization of this paper is as follows. In section II, we construct the Lagrangian and the Wave function for the Caldirola-Kanai Oscillator. Section III focuses on the uncertainty relation, we determine the path integral of the Damped Harmonic Oscillator (DHO) and the wave packet in section IV. Section V gives a brief conclusion.

II. LAGRANGIAN AND THE WAVE FUNCTION OF THE MODIFIED CALDIROLA-KANAI OSCILLATOR

We consider a harmonic oscillator with a time-dependent mass $m(t) = me^{\sin\beta\gamma t}$ and described by the Hamiltonian

$$\mathrm{H}(t) = \frac{p^2}{2m} e^{-\sin(\beta\gamma t)} + \frac{m}{2} e^{\sin(\beta\gamma t)} q^2 \qquad [1]$$

whose mass depends explicitly on time, β, γ are variable parameter and damping factors while p and q are canonically conjugate [1]. Eq. (1) reduces to Caldirola-Kanai Oscillator [1, 13, 14, 15, 16, 17, 18] when exp(sin $\beta\gamma t$) is Taylor expanded to first order in increasing power of $\beta\gamma t$ with the variable parameter $\beta \to 1$. The Lagrangian corresponding to the Hamiltonian in Eq. (1) is given as

$$L = e^{\sin\beta\gamma t}\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2(t)q^2\right]. \tag{2}$$

The classical equation of motion is that of a damped oscillator,

$$\ddot{q}(t) + \beta\gamma\cos\beta\gamma t\dot{q}(t) + \omega^2(t)q(t) = 0 \tag{3}$$

with the Hamiltonian Eq. (1), we see that the damped oscillator is prescribed by the time-dependent Schrödinger equation.

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}_{ck}(t)\Psi(x,t). \tag{4}$$

Several methods [17, 21, 22, 24] have been employed in determining the wave functions of Eq. (4). Following the method introduced by Lewis and Risenfeld for an invariant operator for the general time-dependent Oscillator, whose eigenfuncton is an exact quantum state up to a time dependent phase factor [17, 25], we introduce a pair of linear operators [25]

$$\hat{a}(t) = \frac{i}{\sqrt{\hbar}}\left[\varepsilon^*(t)\hat{p} - m\dot{\varepsilon}^*(t)\hat{q}\right],$$
$$\hat{a}^+(t) = \frac{-i}{\sqrt{\hbar}}\left[\varepsilon(t)\hat{p} - m\dot{\varepsilon}(t)\hat{q}\right], \tag{5}$$

Where $\varepsilon(t)$ is the solution to the classical equation of motion and these operators are required to satisfy the quantum-Liouville-Von Neumann equations

$$\frac{i\hbar\partial}{\partial t}\hat{a}(t) + \left[\hat{a}(t), \hat{H}_{ck}(t)\right] = 0.$$
$$i\hbar\frac{\partial}{\partial t}\hat{a}^+(t) + \left[\hat{a}^+(t), \hat{H}_{ck}(t)\right] = 0 \tag{6}$$

In addition, the quantity $\varepsilon(t)$ must also satisfy the classical damped equations of Eq. (3) and likewise satisfies the Wronskian condition [19].

$$\hbar\exp\left(\frac{d}{d\beta}\sin\beta\gamma t\right)\left[\dot{\varepsilon}^*(t)\varepsilon(t) - \dot{\varepsilon}(t)\varepsilon^*(t)\right] = i.$$

when the variable parameter $\beta \to 1$, this then guarantees the equal-time commutation relation.

$$\left[\hat{a}(t), \hat{a}^+(t)\right] = 1 \tag{8}$$

The general solution of Eq. (3) is given by

$$q(t) = e^{-\beta\gamma t\cos\beta\gamma t}\left[Ae^{i\Omega t} + Be^{-i\Omega t}\right], \tag{9}$$

where

$$\Omega(t) = \left[\omega^2 - \left(\frac{\gamma}{4}\right)^2\right]^{1/2}$$

Setting $A = B$, in Eq. (9), we get;

$$A = \frac{1}{\sqrt{2\hbar\Omega}}, \tag{10}$$

Using Eq. (10), we obtain for the under damped oscillator $\gamma \leq 2\omega$ as

$$\varepsilon(t) = \frac{1}{\sqrt{2\hbar\Omega}}e^{-(\beta\gamma t\cos\beta i\gamma t + i\Omega t)} \tag{11}$$

The solution of quantum damped oscillator in the regime of strong damping $\gamma > 2\omega$ and the critical damping $\omega = 0$ are given by

$$\varepsilon(t) = e^{-\beta\gamma t\cos\beta\gamma t}\left[Ae^{-\Omega t} + Be^{-\Omega t}\right], \tag{12}$$

$$\varepsilon(t) = A + Be^{\frac{-\beta\gamma t\cos\beta\gamma t}{2}}, \tag{13}$$

respectively.

Using Eq. (10), we obtain a generalize form of Eq. (11) as

$$\varepsilon(t) = \frac{e^{-\beta\gamma t\cos\beta\gamma t}}{\sqrt{2\hbar\Omega}}\left[\theta(-t)e^{i\Omega t} + \theta(t)e^{-i\Omega tt}\right], \tag{14}$$

where $\theta(t)$ is the Heaviside step function and the solution of Eq. (14) yields

$$\varepsilon(t) = \frac{e^{\frac{-\beta\gamma t\cos\beta\gamma t}{2}}}{\sqrt{2\hbar\Omega}}e^{-i\Omega t}, t > 0$$
$$\varepsilon(t) = \frac{e^{\frac{-\beta\gamma t\cos\beta\gamma t}{2}}}{\sqrt{2\hbar\Omega}}e^{-i\Omega t}, t < 0, \tag{15}$$

By performing analytic continuation or Wick rotation $t \to it'$ on Eq. (15), we obtain the solution in the regime of strong damping $\gamma > 2\omega$ as

$$\varepsilon(t)=\frac{e^{\frac{-\beta\gamma t\cos\beta\gamma t}{2}}}{\sqrt{2\hbar\Omega}}e^{\Omega t'},t>0$$
$$\varepsilon(t)=\frac{e^{\frac{-\beta\gamma t\cos\beta\gamma t}{2}}}{\sqrt{2\hbar\Omega}}e^{-\Omega t'},t<0, \quad (16)$$

The solution for the critical damping ω = 0, using Eq. (10) gives

$$\varepsilon(t)=\frac{1}{\sqrt{2\hbar\Omega}}\left[1+e^{-\frac{\beta\gamma t\cos\beta\gamma t}{2}}\right], \quad (17)$$

Eqs. (15), (16) and (17) are the general solution for the quantum under, over and critical damped oscillators.

The eigenfunctions of operator equation (5) are the conjugate set of damped oscillator and solving equation (4), we obtain this wave function in the explicit form as [17]

$$\Psi_n(x,t)=\left(\frac{m\Omega e^{\sin\beta\gamma t}}{\pi\hbar}\right)^{1/4}\frac{e^{-i\left(n+1/2\right)\Omega t}}{\sqrt{2^n n!}} \quad (18)$$
$$\times H_n\left(\frac{m\Omega e^{\sin\gamma rt}}{\hbar}q\right)\exp\left[-e^{\sin\beta\gamma t}\left(\frac{m\Omega}{2\hbar}+\frac{ir}{4h}\right)q^{2t}\right],$$

where H_n is the Hermite polynomials.

III. THE UNCERTAINTY RELATION

With equation (18) the wave function of the damped harmonic oscillator, we obtain the dispersion in co-ordinate space for the under damping regime as

$$\langle\Delta q^2\rangle=\hbar^2\varepsilon^*(t)\varepsilon(t)$$
$$=\frac{\hbar}{2\Omega}e^{-\beta\gamma t\cos\beta\gamma t}, \quad (19)$$

and its momentum counterpart is given by

$$\langle\Delta p^2\rangle=\hbar^2 m'\dot{\varepsilon}^*(t)\varepsilon(t)$$

$$=\frac{\Omega}{2\hbar}\left[1+\left(\frac{\beta\gamma\sigma(t)}{2\Omega}\right)^2\right]e^{(2-\beta)\gamma t\cos\beta\gamma t}, \quad)$$

where the reduced mass m′(t) is defined as [19]

$$m'(t)=\exp\left(\frac{d}{d\beta}\sin\beta\gamma t\right), \quad (21)$$

and $\sigma(t)$ is given by

$$\sigma(t)=\left(\cos^2\beta\gamma t-2\beta\gamma t\sin 2\beta\gamma t+\beta^2\gamma^2 t^2\sin^2\beta\gamma t\right)^{1/2}, \quad (22)$$

with the use of equations (19) – (21), we obtain the uncertainty relation easily as

$$(\Delta q\Delta p)=\frac{\hbar}{2}\left[1+\left(\frac{\beta\gamma\sigma(t)}{2\Omega}\right)\right]^{1/2}e^{(1-\beta)\gamma t\cos\beta\gamma t} \quad (23)$$

Equation (23) give a generalized uncertainty relation for the modified Caldirola-Kanai oscillator, it reduces to the Caldirola-Kanai Oscillator when β = 1, and equation (23) becomes

$$(\Delta q\Delta p)=\frac{\hbar}{2}\left[1+\left(\frac{\gamma\sigma'(t)}{2\Omega}\right)^2\right]^{1/2}, \quad (24)$$

where

$$\sigma'(t)=\left[\cos^2\gamma t-2\gamma t\sin 2\gamma t+\gamma^2 t^2\sin^2\gamma t\right]^{1/2} \quad (25)$$

These formulas derived reduces to these of the simple harmonic oscillator (SHO) when γ = 0. Figure 1.0 shows a plot of σ (t) for various damped factor γ = 0, 0.5Ω and 1.0Ω.

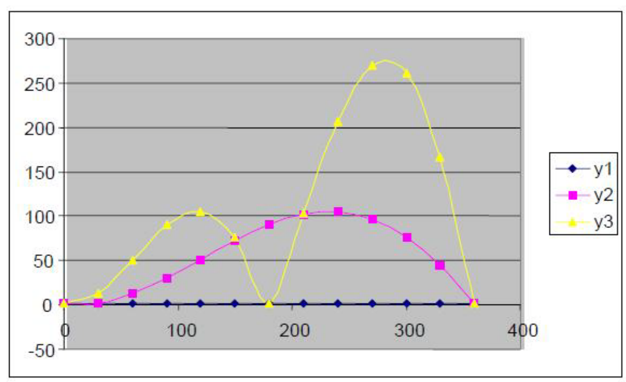

FIGURE 1. Variation of σ(t) with γt for various γ values of 0, 0.5Ω and Ω.

IV PATH INTEGRAL OF THE DAMPED HARMONIC OSCILLATOR AND THE WAVE PACKET

The Path Integral for a particle propagator from the initial point (q_i, t_i) to the final point (q_f, t_f) is given by the integral connecting over all possible paths connecting the initial and the final points.

$$K(q_f,t_f;q_i,t_i)=N\int_{(q_i,t_i)}^{(q_f,t_f)} Dq(t)e^{i/\hbar S[q,q']}; \tag{26}$$

where N is the normalization constant, D(q) the measure and the action is defined as

$$S[q]=\frac{m}{2}\int_{t_i}^{t_i} e^{\sin\beta\gamma t}\left[q^2(t)-\omega^2q^2(t)\right], \tag{27}$$

and decomposing the path into a classical path $q(t)$ and the fluctuation path $g(t)$, we determine the classical path as

$$q_{ck}(t)=\left[\frac{q_f e^{\sin\beta\gamma t_i}\sin\omega(t-t_i)-q_i e^{\sin\beta\gamma t_f}\sin\omega(t_1-t)}{\sin\omega T}\right]e^{-\sin\beta\gamma t}, \tag{28}$$

where $t_f - t_i = T$. On substituting equation (28) into equation (27), we obtain the classical action [18, 19, 23],

$$\begin{aligned}S[q_{cl}]&=\frac{m}{2}\left(\frac{we^{\sin\beta\gamma(t_f+t_i}}{\sin\omega T}\right)\\
&\times\left\{e^{\sin\beta\gamma T}q_f^2(1+\gamma^2\beta^2)\left[\cos\omega T-\frac{\gamma}{2\omega}\sin\omega T\right]\right.\\
&+e^{-\sin\beta\gamma T}q_i^2(1+\gamma^2\beta^2)\left[\cos\omega T+\frac{\gamma}{2\omega}\sin\omega T\right]\\
&\left.-2q_1q_2e^{\sin\beta\gamma(t_i+t_f)}\right\}+\frac{m\gamma\beta}{4}\left(q_i^2e^{\sin\beta\gamma t_i}-q_f^2e^{\sin\beta\gamma t_f}\right)\end{aligned} \tag{29}$$

Substituting equation (29) into equation (26), and evaluating Gaussian (Fresnel) Integral to determine the normalization constant, we obtain the exact form of the propagator for the modified Caldirola-Kanai Hamiltonian of equation (1) as

$$K(q_f,t_f;q_i,t_i)=\sqrt{\frac{m\omega e^{\sin\beta\gamma(t_i+t_f)}}{2\pi i\hbar\sin\omega T}}e^{i/\hbar S[q]} \tag{30}$$

We now make a few remarks about the propagator of equation (30) as follows:

(i) When $\omega\to 0$ and $\beta\gamma\to 0$, the propagator reduces to the free particle propagator.

(ii)

$$K\left(q_f,t_f;q_i,t_i\right)=\left(\frac{m}{2\pi i\hbar(t_f-t_i)}\right)^{1/2}\exp\left[\frac{m}{2\hbar}\frac{(q_f-q_i)^2}{t_f-t_i}\right] \tag{31}$$

(ii) When $\omega\to 0$, the propagation reducing to the propagation of a quasi-free particle with dissipative factor [18]

$$\begin{aligned}K\left(q_f,t_f;q_i,t_i\right)&=\left(\frac{m}{2\pi i\hbar}\right)^{1/2}\left(\frac{\beta\gamma}{e^{-\sin\beta\gamma t_i}-e^{-\sin\beta\gamma t_f}}\right)^{1/2}\\
&\times\exp\left[\frac{im}{2\hbar}(1+\gamma^2\beta^2)\left(\frac{\beta\gamma}{e^{-\sin\beta\gamma t_i}-e^{-\sin\beta\gamma t_f}}\right)\times\left(q_f^2-q_i^2\right)^2\right]\\
&+\frac{m}{4}\left[\left(\frac{\beta\gamma}{e^{-\sin\beta\gamma t_i}-e^{-\sin\beta\gamma t_f}}\right)\left(q_f-q_i\right)^2\right.\end{aligned}$$

(iii) When $\beta\gamma\to 0$, the propagator in this case reduces to the propagator of harmonic oscillator,

$$K(q_f,t_f;q_i,t_i)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega T}\right)^{1/2} \tag{32}$$

$$\times\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[\left(q_f^2+q_i^2\right)\cos\omega T-2q_fq_i\right]\right\} \tag{33}$$

In order to determine the Gaussian wave packet evolving for this propagator, equation (30), we initialize the profile of the packet at $t_i = 0$, and obtain the wave packet as [2, 4, 9, 23].

$$\psi(q_i\ 0)=\left(\frac{1}{2\pi\alpha^2{}_0}\right)^{1/4}\exp(-\frac{(q_i-b)^2}{4\alpha_0^2}), \tag{34}$$

where α_0^2 gives the variance of the Gaussian wave packet and the wave packet is choose to packed at $q_i = b$ at $t_i = 0$. The kernel (propagator), equation (30) satisfies the Schrödinger equation with respect to (q_f, t_f), such that

$$i\hbar\frac{\partial}{\partial t}\langle q_f,t_f\,|\,q_i,t_i\rangle=H(q_f,p_f)\langle q_f,t_f\,|\,q_i,t_i\rangle \tag{35}$$

In general, for any eigenket $|\phi\rangle$, the wave function is given by

$$\psi_f(x,t)=\langle x,t/\varphi), \tag{36}$$

which is the solution of the Schrödinger equation. The wave packet at a time t is related to the wave packet at $t_i = 0$ as

$$\psi(q,t)=\int_{-\infty}^{\infty} dq_i k(q_f,t_f;q_i,0)\psi(q_i,0) \tag{37}$$

On solving Eq. (37), we obtain the square amplitude as [9, 23]

$$\left|\psi(q_f,t)\right|^2=\frac{1}{(2\pi\alpha_t)}^2$$

$$\times\exp\left[-\frac{\left[q_f-b\left(1+\beta^2\gamma^2\right)e^{-\sin\beta\gamma t}\left(\cos\omega T+\frac{\gamma}{2\omega}\sin\omega T\right)\right]^2}{2\alpha_t^2}\right] \tag{38}$$

where

$$\alpha_t^2 = \alpha_0^2\left(1+\beta^2\gamma^2\right)^2 e^{-\frac{1}{2}\sin\beta\gamma t}\left[\left(\cos\omega t+\frac{\gamma}{2\omega}\sin\omega t\right)^2+\left(\frac{\hbar\sin\omega t}{2m\omega\alpha_0^2}\right)^2\right] \quad (39)$$

We observed in equation (38) as was observed by [23] that the wave packet at any time t is packed at

$$q_f = b\left(1+\beta^2\gamma^2\right)e^{-\sin\beta\gamma t}\left(\cos\omega t+\frac{\gamma}{2\omega}\sin\omega t\right). \quad (40)$$

The probability derives $\left|\psi(q_f,t)\right|^2$ in equation (38) has a very similar form to that of equation (23) but our propagator equation (30) takes a new form.

V. CONCLUSION

In conclusion, we have constructed the Lagrangian of the modified Caldirola-Kanai oscillator; we evaluated the uncertainty relation of the damped oscillator (DHO) and show how it reduces to the states of Simple Harmonic Oscillator (SHO). We have also shown how to evaluate the propagator of the harmonic oscillator and then studied the evolution of the Gaussian wave packet arising from the propagator.

VI. ACKNOWLEDGEMENTS

This work was supported by the Imienyong-Nandy and Leabio Research grant No. 46-4756.

REFERENCES

[1] Um, C. I. Choi, J. R., Yeon, K. H. and George, T. F., *Exact quantum theory of the harmonic oscillator in the form of Mathieu functions, Korean Phys. Soc.* **40**, 969 (2002).

[2] Feynman, R. P. and Hibbs, H., *Quantum Mechanics and Path Integrals*, (McGraw Hill, New York, 1965) p. 143

[3] Duru, I. H. and Kleinert, H., *Quantum mechanics of H-Atom from path Integrals*, Fortschr. Phys. **30**, 401 (1982).

[4] Klenert, H., *Path Integrals in Quantum Mechanics, Statistics and Polymers Physics*, (World Scientific Singapore, 1991) p.540

[5] Grosche, C., *Coulomb potentials by path integrals*, Fertschr. Phys. **40**, 695 (1992).

[6] Inomata, A., *Alternative Exact path integrals – Treatment of the hydrogen atom*, Phys. Rev. Lett. A. **101,** 253 (1984).

[7] Ho, R. and Inomata, A., *Exact path integrals solution of the path integrals of Dirac-Coulomb system*, Phys. Rev. Lett. **48**, 231 (1982).

[8] Steiner, F., *Exact path integral Treatment of the H-atom*, Phys. Lett. A. **106**, 363 (1984).

[9] Schulman, L. S., *Techniques and applications of Path Integration*, (John Wiley, New York, 1981) p. 220

[10] Dewitt, C. M., *Semi classical expansion*, Ann. Phys. **97**, 367 (1976).

[11] Dirac, P.A.M., *Quantum Mechanics*, (Wiley, New York, 1963) p.76

[12] Weiss, U., *Quantum dissipative system*, (World Scientific, Singapore, 1993) p. 118.

[13] Kanai, E., *On the quantization of the dissipative systems*, Prog. Theor. Phys. **3**, 440 (1945).

[14] Caldirola, P., *Forze non conservative nella meccanica quantistica*, Nuovo Cim. **18**, 393 (1941).

[15] Um, C. I. and Yeon, K. H., *Quantum theory of the harmonic oscillator in non-conservative systems*, J. Korean Phys. Soc. **41**, 594 (2002).

[16] Um, C. I., Yeon, K. H. and George, T. F., *The Quantum harmonic oscillator*, Phys. Rept. **362**, 63 (2002).

[17] Kim, S. P., Santana, A. E. and Khanna, F. C., *Decoherence of Quantum damped oscillators*, J. Korean Phy. Soc. **43**, 4 (2003).

[18] Huang, M .C. and Wu, M. C., *The Caldirola-Kanai model and its equivalent theories for a damped oscillator*, Chin. J. Phys. **36**, 4 (1998).

[19] Ikot, A. N., Ituen, E. E., Essien, I. E. and Akpabio, L. E., *Path integrals evaluation of a time-dependent oscillator in an external field*, Turk. J. Phys. **32**, 305 (2008).

[20] Lewis Jr., H. R. and Risenfeld, R., *An exact quantum theory of a time-dependent harmonic oscillator and of a charge particle in a time-dependent electromagnetic field*, Math. Phys. **10**, 1458 (1969).

[21] Um, C. I., Kim, I. H., Yeon, K. H., George, T. F. and Pandey, L. N., *Exact quantum theory of a time-dependent bound quadratic Hamiltonian system*, Phys. Rev. A. **54**, 2707 (1996).

[22] Dodonor, V. V., George, T. F., Maniko, O. V., Um, C. I. and Yeon, K. H., *Propagator and wavefunction for a damped oscillator in a resonator with time dependent medium*, J. Sov. Lasers Res. **13**, 219 (1992).

[23] Jain, D., Das, A. and Kar, S., *Path integrals and wavepacket evolution for damped mechanical systems*, arxiv: quant-Ph/0611239V1, 2006.

[24] Khandekar, D. C. and Lawande, S. V., *Exact solution of a time-dependent quantal harmonic oscillator with damping and a perturbative force*, J. Math. Phys. **20**, 1870 (1979).

[25] Kim, S. P., *Time-dependent displaced and squeezed number states*, J. Phys A, **36**, 12089 (2003).