A heuristic review of Lanczos potential

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Abstract

We present a review of 3-tensor potential L_{abc} proposed by Lanczos for the Weyl conformal curvature tensor. We show the role that plays Lanczos tensor in General Relativity for theoretical physics postgraduate students. In the same way that the electromagnetic vector potential can be used to compute the Maxwell field, the Lanczos potential can also be employed to compute the Weyl curvature tensor of a gravitational field.

Key words: Lanczos potential theory, Weyl-Lanczos equations, Lanczos coefficients.

Resumen

Realizamos una revisión del potencial 3-tensorial propuesto por Lanczos L_{abc} para el tensor de Weyl de curvatura conforme. Mostramos el papel que juega el tensor de Lanczos en Relatividad General para estudiantes de posgrado de física teórica. En la misma forma que el vector potencial puede ser usado para calcular el campo de Maxwell, el potencial de Lanczos también puede ser empleado para calcular el tensor de curvatura de Weyl de un campo gravitacional.

Palabras clave: Teoría de potenciales de Lanczos, ecuaciones de Weyl-Lanczos, coeficientes de Lanczos.

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I. INTRODUCTION

In the recent years there has been a renewed interest in the 3-tensor potential L_{abc} proposed by Lanczos for the Weyl curvature tensor [1]. However, in the General Relativity and gravitation most popular textbooks, like Misner, Thorne and Wheeler [2], Wald [3], Hawkings *et al.* [4], Weinberg [5], Penrose and Rindler [6], Stephani *et al.* [7], etc. there is not any treatment about Lanczos potential theory. Nevertheless, it is important in theoretical and physical aspects because we acknowledge that Einstein equations can be written in terms of the covariant derivative of Lanczos Potential in its Jordan form; also, there is a possibility of the existence of an Aranov-Bhom's gravitational quantum equivalent effect to the traditional one [8]. In the educational field, the importance of Lanczos potential is clear because of its analogy with the electromagnetic 4-vector potential.

Our aim in this work is to present an heuristic point of view of Lanczos potential that reaffirms the last mentioned analogy between gravity and electromagnetism. This is focused to postgraduate General Relativity students. The paper is organized as follows: in Sect. II, we present some algebraic properties of Lanczos potential; in Sect. III, we mentioned the physical interpretation of Lanczos potential; en Sect. IV, we show the method for the vacuum space-times; in Sect. V, we show a brief example for Schwarzschild space-time. Finally, in Sect. VI, we present our conclusions. Appendix A.I is devoted to spinor formalism.

II. ALGEBRAIC PROPERTIES OF LANCZOS POTENTIAL

In General Relativity, we often need to work in a given space-time, or to derive one in the form of an exact solution to Einstein's gravity equations. In 1962 Lanczos suggested an auxiliary potential [9]; through it and the covariant derivative, we can obtain the conformal curvature C_{abcd} of decomposition of Riemann curvature tensor

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd}, \qquad (1)$$

where the following abbreviations are followed

$$E_{abcd} \equiv \frac{1}{2}(g_{ac}S_{bd} + g_{ad}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}),$$

$$G_{abcd} \equiv \frac{R}{12}(g_{ac}g_{bd} - g_{ad}g_{bc}) \equiv \frac{R}{12}g_{abcd}, \quad (2)$$

$$S_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}, \qquad R \equiv R^{a}{}_{a}.$$

The Lanczos potential is studied in relation to a Weyl candidate, *i.e.,* a 4-rank tensor with the following expression

$$\begin{aligned} W_{abcd} &= L_{abc;d} - L_{abd;c} + L_{cdb;a} - L_{cda;b} \\ &+ L_{(ad)}g_{bc} + L_{(bc)}g_{ad} - L_{(ac)}g_{bd} \\ &- L_{(bd)}g_{ac} + \frac{2}{3}L^{rs}{}_{r;s}(g_{ac}g_{bd} - g_{ad}g_{bc}), \end{aligned} \quad (3)$$

where we define

$$L_{ad} \equiv L_{a}{}^{r}{}_{d;r} - L_{a}{}^{r}{}_{r;d}. \quad (4)$$

The above equation is equivalent to the dual form

$$\begin{aligned} W_{abcd} &= L_{ab[c;d]} + L_{cd[a;b]} \\ &- {}^{*}L^{*}_{ab[c;d]} - {}^{*}L^{*}_{cd[a;b]}. \end{aligned} \quad (5)$$

Initially, an arbitrary tensor of third order has 4^3 free components, but further we have to impose the following 40 algebraic symmetries

$$L_{abc} = -L_{bac}, \quad (6)$$

the four *(gauge algebraic conditions of Lanczos)*

$$L_{a}{}^{r}{}_{r} = 0, \quad (7)$$

and the dual four conditions

$$^{*}L_{a}{}^{r}{}_{r} = 0, \quad (8)$$

or equivalently

$$L_{abc} + L_{bca} + L_{cab} = 0, \quad (9)$$

then, his initially sixty four degrees of freedom are reduced to sixteen. Further six *differential Lanczos gauge conditions*

$$L_{ab}{}^{r}{}_{;r} = 0. \quad (10)$$

In fact, the Lanczos potential also admits a wave equation [2], the tensorial form of this wave equation has proved to be very useful to find Lanczos potential by some interesting methods due to Velloso and Novello [3]. The complete form of the tensorial equation for Lanczos potential is

$$\begin{aligned} \Box L_{abc} &+ 2R_{c}{}^{r}L_{abr} - R_{a}{}^{r}L_{bcr} - R_{b}{}^{r}L_{car} \\ &- g_{ac}R^{rs}L_{rbs} + g_{bc}R^{rs}L_{ras} - \frac{1}{2}RL_{abc} = J_{abc}. \end{aligned} \quad (11)$$

III. PHYSICAL INTERPRETATION OF LANCZOS POTENTIAL

The role of Lanczos potential L_{abc} with respect to the Weyl tensor W_{abcd} is the same that plays the vector potential A_a for the Maxwell tensor F_{ab}. Then, in the same way that we derive the electromagnetic tensor F_{ab} from A_a, we can derive the electromagnetic field F_{ab} as:

$$F_{ab} = W_{ab}(A) = A_{a;b} - A_{b;a}, \quad (12)$$

we can also derive the Weyl curvature tensor from a potential; nevertheless, this could not be a vector potential as in the electromagnetic case, instead it must be a 3-rank tensor called *the Lanczos potential* L_{abc} , with similar properties, see equation (3).

Also, assuming that some space-time M admits an electromagnetic field F_{ab} = - F_{ba} , then this field obeys certain rules. For example

$$\begin{aligned} F_{ab;c} + F_{bc;a} + F_{ca;b} &= 0, \\ F^{cd}{}_{;d} &= J^{c}. \end{aligned} \quad (13)$$

Therefore, the gravitational Lanczos potential L_{abc} physically corresponds with the electromagnetic vector potential A_a. In a certain point of view, A_a could be derived in a covariant way to get the electromagnetic tensor F_{ab} (as is suggested by (12)). Also, the Lanczos potential could be derived in a covariant way to get the Weyl gravitational field as it is correspondingly suggested by (12). The source of the electromagnetic equation (second equation of (13)) corresponds to (11) Lanczos equation with sources.

IV. METHOD FOR THE VACUUM SPACE-TIMES

If we have a space-time with a global Killing vector field ξ^a [11], it is sometimes possible to generate a Lanczos potential with the following method:

If ξ^a is a non-null Killing vector that satisfies Killing equation

$$\xi_{a;b} + \xi_{b;a} = 0, \quad (14)$$

and also is hyper surface orthogonal vector (*i.e.* satisfies the following mathematical relation)

$$\xi_{[a;b}\xi_{c]} = 0. \quad (15)$$

Then, it is possible to take an unit vector u_a from the group of motions, such that

$$u_a = \frac{\xi_a}{\xi}, \quad \xi^2 = s\xi_a\xi^a > 0, \quad s = \begin{cases} 1 \\ -1 \end{cases}. \quad (16)$$

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Thus, the Killing equation (14) guarantees that u_a is *expansion-less* and *shear-free, i.e.*

$$u_{(a;b)}(\delta_m{}^a - u^a u_m)(\delta_n{}^b - u^b u_n) = 0, \quad (17)$$

and by means of the hyper surface orthogonallity condition of ξ^a we have

$$u_{[a;b}u_{c]} = 0, \quad (18)$$

therefore,

$$u_{a;b} = sa_a u_b, \quad (19)$$

where we have defined the first curvature vector of the group of congruence (also called and known as acceleration) to be $a_a=u_{a;b}u^b$ which for all group of motions even those not satisfying (18), this is a gradient $a_a=(\ln(1/\xi))_{,a}$. Then, a candidate of Lanczos potential is given by

$$L_{abc}=(a_a u_b - a_b u_a)u_c - \frac{1}{3}s(a_a g_{bc} - a_b g_{ac}), \quad (20)$$

which satisfies (6), (7) and (9). Then, verifying the Lanczos gauge (10) and the (3) condition of the potential, we can fix completely our candidate as a full-fledged Lanczos potential. We can trace the spinorial analog of (3) to fix our candidate. In the following section, we show an example for the Schwarzschild space-time.

V. AN EXAMPLE FOR THE SCHARZSCHILD SPACE-TIME.

Now, we show a brief example of the method [11] using the Schwarzschild line element in the coordinates, *(t, r, θ, ø)* as follows

$$ds^2=\left(1-\frac{2M}{r}\right)dt^2-\frac{dr^2}{1-\frac{2M}{r}} - r^2\, d\theta^2 - r^2\sin(\theta)^2\, d\phi^2. \quad (21)$$

The time-like Killing vector $\xi^a=\delta^a_0$ has squared norm

$$\xi^2=g_{00}=\left(1-\frac{2M}{r}\right), \quad (22)$$

clearly this vector fields are hypersurface-orthogonal. If we introduce the corresponding unit vector field,

$$u_a=\frac{\xi_a}{\xi}, \quad u_a u^a=1, \quad (24)$$

then

$$u_{a;b}=a_a u_b, \quad \text{with} \quad a_a=u_{a;b}u^b, \quad (25)$$

now, with the velocity vector u_a and the acceleration a_a, it is possible to write a Lanczos potential in the following form

$$L_{abc}=(a_a u_b - a_b u_a)u_c - \frac{1}{3}(a_a g_{bc} - a_b g_{ac}), \quad (26)$$

this is the 3-rank tensor that Novello and Velloso [10] prove to be a Lanczos potential. We start to manage the example by using a preferred null tetrad in the Newmann-Penrose convention, say

$$\{e_a\}=(m_a,\overline{m}_a,l_a,k_a),$$

$$g_{ab}=2m_{(a}m_{b)}-2k_{(a}l_{b)}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}, \quad (27)$$

$$m_a\overline{m}^a=1, \quad k_a l^a=-1,$$

we continue the present example employing the null tetrad provided by Kinnersley, which are

$$l^a=\frac{1}{\Delta}(r^2,\Delta,0,0), \quad n^a=\frac{1}{2r^2}(r^2,-\Delta,0,0),$$
$$\Delta\equiv r^2-2Mr,$$
$$m^a=\frac{1}{\sqrt{2}r}(0,0,1,i\csc(\theta)),$$
$$\overline{m}^a=\frac{1}{\sqrt{2}r}(0,0,1,-i\csc(\theta)). \quad (28)$$

Here, the non-vanishing spin coefficients and Weyl scalar components are

$$\rho=-\frac{1}{r}, \quad \beta=-\alpha=\frac{\cot(\theta)}{2\sqrt{2}r}, \quad \mu=-\frac{r-2M}{2r^2},$$
$$\gamma=\frac{M}{2r^2}, \qquad \Psi_2=\frac{M}{r^3}, \quad (29)$$

then, the intrinsic derivatives are

$$D=l^a\nabla_a=\frac{r^2}{\Delta}\frac{\partial}{\partial t}+\frac{\partial}{\partial r},$$
$$D'=n^a\nabla_a=\frac{1}{2}\frac{\partial}{\partial t}-\frac{\Delta}{2r^2}\frac{\partial}{\partial r},$$
$$\delta=m^a\nabla_a=\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi}\right),$$
$$\delta'=\overline{m}^a\nabla_a=\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi}\right), \quad (30)$$

from these set of relations we have that the only non-vanishing components of Lanczos tensor are

$$L_1=-\frac{1}{3}\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}, \quad L_6=-\frac{M}{6r^2}, \quad (31)$$

where we used the following relations between the null tetrad and the Weyl scalars

$$\Psi_0=C_{abcd}l^a m^b l^c m^d, \quad \Psi_1=C_{abcd}l^a m^b l^c n^d,$$
$$\Psi_2=C_{abcd}l^a m^b \overline{m}^c n^d, \quad \Psi_3=C_{abcd}l^a n^b \overline{m}^c n^d,$$
$$\Psi_4=C_{abcd}\overline{m}^a n^b m^c n^d, \quad (32)$$

and Lanczos scalars

$$L_0=L_{abc}l^a m^b l^c, \quad L_1=L_{abc}l^a m^b \overline{m}^c,$$
$$L_2=L_{abc}\overline{m}^a n^b l^c, \quad L_3=L_{abc}\overline{m}^a n^b m^c,$$
$$L_4=L_{abc}l^a m^b m^c, \quad L_5=L_{abc}l^a m^b n^c,$$
$$L_6=L_{abc}\overline{m}^a n^b m^c, \quad L_7=L_{abc}\overline{m}^a n^b n^c. \quad (33)$$

This is an example of a Lanczos potential that has been computed from the Novello and Velloso's Lanczos gravitational potential 3-rank tensor that illustrates the use of the null tetrad and spinorial coefficients from the Newmann- Penrose formalism [10].

VI. CONCLUSIONS

It has been found that the method of Novello and Velloso [10] described in section IV, could be used to compute a Lanczos gravitational potential, and it has been pointed out that we can use it in the derivation of the Weyl curvature tensor for the simple case of a Schwarzschild spherically symmetric vacuum space-time. The Lanczos tensor is important in the derivation of the Einstein field equations in his Jordan form. We have used the spin-coefficients and the Newmann-Penrose formalism in this method to achieve our objective of showing an easy and convenient way to get this important quantity.

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REFERENCES

[1] Bergqvist, G., *A Lanczos potential in Kerr geometry*, J. Math. Phys. **38**, 3142-3154 (1997); Edgar, S. B. and Hoglund, a., *The Lanczos potential for the Weyl curvature tensor: existence, wave equation and algorithms*, Proc. Roy. Soc. Lond. **A 453**, 835-851 (1997); López-Bonilla, J. L., Ovando, G. and Peña, J. J., *Lanczos potential for plane gravitational waves*, Found. Phys. Lett. **12**, 401-405 (1999); Cartin, D., *The Lanczos potential as a spin-2* field, hep-th/0311185v1 (2003); O'Donnell, P., *Lanczos tensor potential for conformally flat space*-times **119**, 341-345 (2004); Mena, F. C and Tod, P., *Lanczos potentials and a definition of gravitational entropy for perturbed FLRW space-times*, gr-qc/0702057v1 (2007).

[2] Misner, Ch. W., Thorne, K. S. and Wheeler, J. A. (*Gravitation*, Freeman, New York, 1995).

[3] Wald, Robert M. (*General Relativity*, The University of Chicago Press, Chicago and London, 1984).

[4] Hawkings, S. W., Ellis, G. F. R. (*The Large Scale Structure of Space-Time*, Cambridge Universty Press, Cambridge, 1973).

[5] Weinberg, S. (*Gravitation and Cosmology: Principles and Aplications of the General Theory of Relativity*, John Wiley, New York, 1972).

[6] Penrose, R. and Rindler, W. (*Spinors and space-time, vol. 1 "Two-Spinor Calculus and Relativistic Fields"*, Cambridge University Press, Cambridge, 1984).

[7] Stephani, H., Kramer, D., Maccallum, M., Hoenselaers, C., Herlt, E. (*Exact Solutions of Einstein's Field Equations,* Cambridge University Press, Cambridge, 2006).

[8] Holgersson, D. (*Lanczos potentials for perfect fluid cosmologies,* Linköping University Thesis, 2004); on-line at: http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-2582

[9] Lanczos, C., *The splitting of the Riemann tensor*, Rev. Mod. Phys. **34**, 379-389 (1962).

[10] Novello, M. and Velloso, A. L., *The connection between general observers and Lanczos potential*, Gen. Rel. Grav. **19**, 1251-1265 (1987).

[11] Dolan, P. and Kim, C. W., *Some solutions of the Lanczos vacuum wave equation*, Proc. R. Soc. Lond. **447**, 577-585 (1994).

A.I NEWMANN-PENROSE CONVENTION FOR THE SPIN-COEFFICIENTS

In this section we show the symbology employed by Newmann and Penrose to denote several spin-coefficients that we have already computed for the vacuum space-time. This notation has been proved to be very useful. The Christoffel symbols of second kind can be written in terms of spinorial coefficients as

$$\Gamma_{ab}{}^{c} = \gamma_{A\dot{A}B}{}^{C}\varepsilon_{\dot{B}}{}^{\dot{C}} + \overline{\gamma}_{\dot{A}A\dot{B}\dot{C}}\varepsilon_{B}{}^{C}, \tag{34}$$

where we followed the convention of decomposition of the null tetrad in terms of the canonical spinorial frame

$$\begin{aligned} m^{a} &= o^{A}\overline{\iota}^{\dot{A}}, \quad \overline{m}^{a} = \iota^{A}\overline{o}^{\dot{A}}, \\ l^{a} &= \iota^{A}\overline{\iota}^{\dot{A}}, \quad k^{a} = o^{A}\overline{o}^{\dot{A}}, \end{aligned} \tag{35}$$

the basis spinors o^{A} and ι^{A} from a *dyad* in the sense that satisfies the normalized relation of inner product:

$$o_{A}\iota^{A} = \varepsilon_{AB}o^{A}\iota^{B} = 1,$$

and ε_{AB} is the "local metric" of the spin space SL(2,C**),** which has the components

$$\varepsilon_{AB} = \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The spinor indices are raised and lowered according to the rules

$$\alpha^{A} = \varepsilon^{AB}\alpha_{B}, \quad \alpha_{A} = \varepsilon_{AB}\alpha^{B}. \tag{36}$$

The spin inner product is antisymmetric in the sense that the spinorial indices *A, B,…* do not commute, *i.e.*

$$\begin{aligned} \alpha_{A}\varepsilon^{AB} &\neq \varepsilon^{BA}\alpha_{A}, \\ \varepsilon_{AB}\alpha^{A}\beta^{B} &= \alpha_{A}\beta^{A} = -\alpha^{A}\beta_{A}. \end{aligned} \tag{37}$$

The diagrams of Newmann-Penrose are the following

TABLE I. Symbology of Newmann-Penrose for the spinor $\gamma_{A\dot{A}B}{}^{C}$.

$A \quad \dot{A} \setminus B \quad C$	$0 \quad 0$	$0 \quad 1$	$0 \quad 1$	$1 \quad 1$
$0 \quad \dot{0}$	ε	$-\kappa$	$-\tau'$	γ'
$1 \quad \dot{0}$	α	$-\rho$	$-\sigma'$	β'
$0 \quad \dot{1}$	β	$-\sigma$	$-\rho'$	α'
$1 \quad \dot{1}$	γ	$-\tau$	$-\kappa'$	ε

TABLE II. Symbology of Newmann-Penrose for the spinor $\gamma_{A\dot{A}BC}$.

$A \quad \dot{A} \backslash B \quad C$	0 0	1 0=1 0	1 1
$0 \quad \dot{0}$	κ	$\varepsilon=-\gamma'$	$\pi=-\tau'$
$1 \quad \dot{0}$	ρ	$\alpha=-\beta'$	$\lambda=-\sigma'$
$0 \quad \dot{1}$	σ	$\beta=-\alpha'$	$\mu=-\rho'$
$1 \quad \dot{1}$	τ	$\gamma=-\varepsilon'$	$\nu=-\kappa'$

There are also two more tables for the complex conjugate spin-coefficients symbols.

TABLE III. Symbology of Newmann-Penrose for the spinor $\gamma_{A\dot{A}\dot{B}}{}^{\dot{C}}$.

$A \quad \dot{A}\backslash\dot{B} \quad \dot{C}$	$\dot{0} \quad \dot{0}$	$\dot{0} \quad \dot{1}$	$\dot{0} \quad \dot{1}$	$\dot{1} \quad \dot{1}$
$0 \quad \dot{0}$	$\bar{\varepsilon}$	$-\bar{\kappa}$	$-\bar{\tau}'$	$\bar{\gamma}'$
$0 \quad \dot{1}$	$\bar{\alpha}$	$-\bar{\rho}$	$-\bar{\sigma}'$	$\bar{\beta}'$
$1 \quad \dot{0}$	$\bar{\beta}$	$-\bar{\sigma}$	$-\bar{\rho}'$	$\bar{\alpha}'$
$1 \quad \dot{1}$	$\bar{\gamma}$	$-\bar{\tau}$	$-\bar{\kappa}'$	$\bar{\varepsilon}'$

TABLE IV. Symbology of Newmann-Penrose for the spinor $\gamma_{A\dot{A}\dot{B}\dot{C}}$.

$A \quad \dot{A}\backslash\dot{B} \quad \dot{C}$	$\dot{0} \quad \dot{0}$	$\dot{0} \quad \dot{1}=\dot{1} \quad \dot{0}$	$\dot{1} \quad \dot{1}$
$0 \quad \dot{0}$	$\bar{\kappa}$	$\bar{\varepsilon} = -\bar{\gamma}'$	$\bar{\pi} = -\bar{\tau}'$
$0 \quad \dot{1}$	$\bar{\rho}$	$\bar{\alpha}=-\bar{\beta}'$	$\bar{\lambda} = -\bar{\sigma}'$
$1 \quad \dot{0}$	$\bar{\sigma}$	$\bar{\beta}=-\bar{\alpha}'$	$\bar{\mu}=-\bar{\rho}'$
$1 \quad \dot{1}$	$\bar{\tau}$	$\bar{\gamma} = -\bar{\varepsilon}'$	$\bar{\nu}=-\bar{\kappa}'$

Notice that in these tables the second and third rows, of the complex conjugate of spin-coefficient $\gamma_{A\dot{A}B}{}^{C}$ and $\gamma_{A\dot{A}BC}$ are interchanged. Because of the obvious relations

$$\overline{\gamma_{A\dot{A}B}{}^{C}} = \overline{\gamma}_{\dot{A}A\dot{B}}{}^{\dot{C}} = \overline{\gamma}_{A\dot{A}\dot{B}}{}^{\dot{C}}, \tag{38}$$

and correspondingly

$$\overline{\gamma_{A\dot{A}BC}} = \overline{\gamma}_{\dot{A}A\dot{B}\dot{C}} = \overline{\gamma}_{A\dot{A}\dot{B}\dot{C}}. \tag{39}$$

(We also can think that the capital latin spinorial indices with dot $\dot{A}, \dot{B}, \ldots$ can not *see* the spinorial indices without a dot $A, B, \ldots$ and then, can permute with this last set without *interference.* But the capital indices without the dot $A, B, \ldots$ *can not* commute between them, because in doing so, they could *interfere* with the value of the spinorial quantity.)