Periodic motions: How their period changes with amplitude of the oscillations and the friction?

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Abstract
The pendulum is a classical experiment, easy to perform and produces data with very simple equipment. The change in the period with the amplitude appears even with small amplitudes if you have some precision in the period. To measure amplitudes larger than 90° we need to change the cord for a rigid rod. We found that for a cylindrical magnet hanging horizontally in the Earth magnetic field, the movement equation is similar to that of a pendulum. We measured amplitudes and times so we can fit a numerical solution to the equation. The initial amplitude is near to 162° and we found that the change in the period comes mainly from the non-linear terms in the equation, and that the friction (either proportional to the angular velocity or its square) only reduces the amplitude with time. With periods from 9 to 4 seconds, the measurements with a protractor can be made manually for many oscillations. For the numerical calculations *Δt* needs to be 0.0001 seconds to conserve the energy.

Key words: Laboratory experiments and apparatus, Numerical simulation studies, Rotational dynamics.

Resumen
El péndulo es un experimento clásico, fácil de realizar y proporciona datos con equipo muy simple. El cambio en el período con la amplitud se nota aún con amplitudes pequeñas si se mide el período con cierta precisión. Medir amplitudes mayores de 90° requiere utilizar una barra rígida en lugar de una cuerda. Encontramos que para un imán cilíndrico colgado horizontalmente en el campo magnético terrestre, la ecuación de movimiento es similar a la de un péndulo. Medimos amplitudes y tiempos para ajustar una solución numérica. La amplitud inicial es cercana a 162° y encontramos que el cambio en el período proviene de la no-linealidad de la ecuación, y que la fricción (proporcional a la velocidad o a su cuadrado) solo cambia la amplitud con el tiempo. Con períodos de 9 a 4 segundos el medir con un transportador se puede hacer manualmente para muchas oscilaciones. Para el cálculo numérico se necesita que *Δt* sea 0.0001 segundos para que se conserve la energía.

Palabras clave: Aparatos y experimentos de laboratorio, estudios de simulación numérica, dinámica rotacional.

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I. INTRODUCCIÓN

The periodic motion exhibited by a simple pendulum is harmonic only for small angle oscillations [1]. Beyond this limit, the equation of motion is nonlinear. We use data from a very simple experiment, to illustrate how to solve numerically those nonlinear equations. Many of the problems to be solved by our students in real life required numerical solutions, so we need to teach them practical hints about the size of the required time interval or fitting parameters graphically. We found that the change in the period comes mainly from the non-linear terms in the equation, and that the friction (either proportional to the angular velocity or its square) only reduces the amplitude with time. We will mention some of the previous work on the topic, and then the analytical solutions to the harmonic oscillator including a friction term proportional to the speed, the pendulum with a period amplitude dependent, the torsion pendulum with the same solution as the harmonic oscillator and a torsion pendulum made with a horizontal magnet in the Earth magnetic field. The measured data will come from this last system.

This paper is organized as follows. In Sect. II we present the basic theory of the harmonic oscillator, pendulum, torsion pendulum, and a magnet in a magnetic field. Also we describe our experimental setting and present some of our results. By using numerical methods in Sect. III we take into account the friction. Finally, in Sect. IV we discuss our main results.

II. THEORY

Although an integral expression exists for the period of the nonlinear pendulum, it is usually not discussed in introductory physics classes because it is not possible to evaluate the integral exactly in terms of elementary functions [2]. Simple approximate expression has been derived for the dependence of the period of a simple pendulum on the amplitude [3,4]. There are several measurements with enough precision to detect the change in the period with the amplitude [5,6], and with angular amplitudes [7,8] near 70°, even the Fourier transform has been fitted [9] and amplitudes near 160° has been measured [10] for a rigid rod pendulum. An oscillating magnet in the magnetic field of a coil has been measured [11] for a few oscillations.

A. Harmonic oscillator

The classical equation for a harmonic motion, a mass *m* supported by a spring of elastic constant *k*, is:

$$md^2x/dt^2 = -kx\,, \tag{1}$$

with a solution:

$$x = A\sin(\omega t + \beta), \tag{2}$$

where *A* is the amplitude of the periodic movement, the angular frequency $\omega = (k/m)^{1/2}$ and β the initial angular phase. The period $T = 2\pi(m/k)^{1/2}$ and amplitude *A* are constants. A more realistic model introduces a friction force proportional to the velocity (λv) so the equation is:

$$md^2x/dt^2 = -\lambda dx/dt - kx, \tag{3}$$

where λ is the friction coefficient. The analytical solution is:

$$x = Ae^{-\gamma t}\sin(\omega' t+\beta), \tag{4}$$

where $\gamma = \lambda/2m$, so the equation predicts that the amplitude decays exponentially with the time and the angular frequency ω' is given by [1]:

$$\omega' = (\omega^2 - \gamma^2)^{1/2}, \tag{5}$$

where ω is the calculated frequency without friction.

The time constant $\tau = 1/\gamma$ measures how fast the amplitude decays, if $t = \tau$ the amplitude decays to $A/e = 0.37A$. If $\omega = 1$ and $\tau = 100$ s, then $\gamma = 0.01$ and we need to measure ω' with 5 digits to detect the change in the period. The decaying amplitude allows an easy measurement of the friction coefficient.

B. Pendulum

Applying Newton's Second Law to the movement of a pendulum on an inextensible cord, a punctual mass and without friction we obtain:

$$l(d^2\theta/dt^2) = -g\sin\theta, \tag{6}$$

where θ is the angle from the vertical, *l* is length of the pendulum and *g* the acceleration of gravity.

In the approximation valid for small amplitudes ($\sin\theta \approx \theta$), the equation becomes that of a harmonic oscillator ($d^2\theta/dt^2 = -g\theta/l$) with a constant period $T = 2\pi\sqrt{l/g}$, independent of the amplitude of the movement. Analytically the equation produces a solution in a series: $T = T_0\,(\,1 + (1^2/2^2)^*\ sen^2(\theta_0/2) + (1^2/2^2)^*(3^2/4^2)^*sen^4(\theta_0/2) + \ldots)$ where θ_0 is the amplitude and T_0 is the period for the harmonic oscillator ($2\pi(l/g)^{1/2}$) [12]. Table I shows the calculated values T/T_0 for values from 1° to 25° for the amplitude of the movement for 1 and 2 terms of the series.

TABLE I. Period T (normalized to T_0) as a function of the amplitude of the oscillation for 1 and 2 terms of the series.

Angle °	*T/To* 1term	T/To 2terms
1	1.000019	1.000019
2	1.000076	1.000076
3	1.000171	1.000171
4	1.000304	1.000305
5	1.000476	1.000476
6	1.000685	1.000686
7	1.000932	1.000934
8	1.001216	1.001220
9	1.001539	1.001544
10	1.001899	1.001907
11	1.002297	1.002308
12	1.002732	1.002748
13	1.003204	1.003227
14	1.003713	1.003744
15	1.004259	1.004300
16	1.004842	1.004895
17	1.005462	1.005529
18	1.006118	1.006202
19	1.006810	1.006915
20	1.007538	1.007666
21	1.008302	1.008458
22	1.009102	1.009288
23	1.009937	1.010159
24	1.010807	1.011070
25	1.011712	1.012020

To measure those small changes we need precision on the measurements, with 0.1% (three digits) we should see the change in the period from 8° of amplitude, but for 1% (two digits) is from 23°-24°. Usually the students uses amplitudes near 30° and measure with three digits, they detect the change neglecting it because the book says that the period is a constant!

C. Torsion Pendulum

For a wire supporting an object with an inertia moment *I*, the applied torque τ is proportional to the torsion angle θ ($\tau = -k\theta$) where *k* is the torsion elastic constant.

In this case the equation for the movement is:

$$I(d^2\theta/dt^2) = -k\,\theta, \qquad (7)$$

corresponding to an harmonic oscillator, with a period $T = 2\pi\sqrt{I/k}$ independent of the amplitude. Usually a copper wire can be twisted several turns within the elastic region. With a cooper wire, it is easy to test this result, the period is a constant.

D. Magnet in a Magnetic Field

For a cylindrical magnet (with a magnetic dipole moment ***M***) supported by a long thin cord, when we can neglect the torque coming from the cord compared with the torque provided by the Earth magnetic field **B** ($\boldsymbol{\tau} = \boldsymbol{M}\times\boldsymbol{B}$). Then:

$$I(d^2\theta/dt^2) = -MB\sin\theta, \qquad (8)$$

with a constant period $T = 2\pi\sqrt{I/MB}$ for small values for the amplitude. Theoretically the series solution for the pendulum may be applied here and the maximum amplitude is near 180°. For a pendulum with a cord the maximum amplitude is 90°, so it provides a better experiment to verify the change in the period with the amplitude of oscillation.

We made the experiment using 12 small cylindrical magnets together forming a long magnet. It hangs horizontally from a thin cord trapped between the 6 and 7 magnets (Fig.1). Then we have really a compass oriented in the north-south direction, with a protractor aligned with the supported cord below the magnet it is easy to measure the maximum amplitude of many oscillations. The periods change from 9 to 4 seconds giving enough time to write the amplitude and its time. The period for each oscillation is the difference in time between two consecutive measurements. Table II shows the measured values for the amplitude, its time and the calculated period. Friction reduces the measured amplitudes.

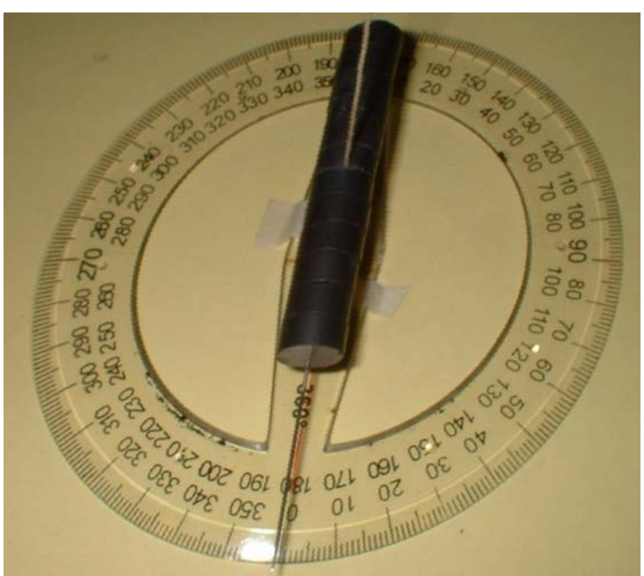

FIGURE 1. Hanging magnet above protractor.

TABLE II. Amplitude of oscillation for a horizontal magnet moving in the Earth magnetic field. The period is the difference in time between two consecutive maximum amplitudes. Uncertainties are ± 1° y ± 0.1 s for the amplitude and time. The chronometer gives a resolution of 0.01 s, but operated manually the repeatability of the operating finger is about 0.1 s. The measured period is too noisy, so we used a moving average over 3 periods for Figure 2 and estimated a maximum error of 1 s for fitting purposes.

Ángle °	Time s	Period s	Ángle °	Time s	Period s
162	7.53	—	88	203.49	5.50
155	16.20	8.67	88	209.34	5.85
150	24.16	7.96	87	213.88	4.54
146	31.66	7.50	86	220.64	6.76
142	38.97	7.31	85	224.35	3.71
139	46.10	7.13	84	228.50	4.15
136	52.89	6.79	83	233.44	4.94
133	58.45	5.56	82	239.84	6.40
131	65.95	7.50	81	245.02	5.18
128	72.44	6.49	80	250.18	5.16
125	78.77	6.33	79	255.45	5.27
122	84.99	6.22	78	260.30	4.85
121	91.15	6.16	76	265.20	4.90
119	97.25	6.10	75	270.63	5.43
116	103.48	6.23	74	275.4	4.77
115	109.67	6.19	73	280.79	5.39
114	115.57	5.90	72	285.6	4.81
110	121.18	5.61	71	290.55	4.95
109	126.75	5.57	70	295.21	4.66
107	132.55	5.80	70	300.42	5.21
106	138.55	6.00	69	305.79	5.37
104	143.91	5.36	68	310.32	4.53
102	149.92	6.01	67	315.00	4.68
101	154.76	4.84	66	320.14	5.14
99	160.31	5.55	65	325.63	5.49
98	165.95	5.64	64	330.08	4.45
97	171.48	5.53	64	335.96	5.88
95	176.85	5.37	63	339.5	3.54
94	182.73	5.88	62	344.87	5.37
92	187.49	4.76	61	349.81	4.94
91	192.81	5.32	60	354.79	4.98
90	197.99	5.18	59	359.40	4.61

From Figure 2 we see that we need up to 16 terms to obtain agreement between the experimental and calculated values without any friction. But we see the friction on the decaying values for the amplitude. The analytical solution for a harmonic oscillator (with a period independent of the amplitude) with a friction force proportional to the velocity of the movement ($f_f = -\lambda v$) is given by Eq. (4), the equation predicts that the amplitude decays exponentially with time. Figure 3 shows an exponential curve fitted to the data with $\gamma = 0.0027$ second^{-1}, a small change in γ produces a better fit on the beginning or on the end of the data. To improve the model we need either a γ changing with the speed or a friction force proportional to the square of the velocity (or other exponent near to 2). The friction parameter is wrong because the theory assumes that the period is independent of the amplitude, approximation not valid for the large amplitudes measured.

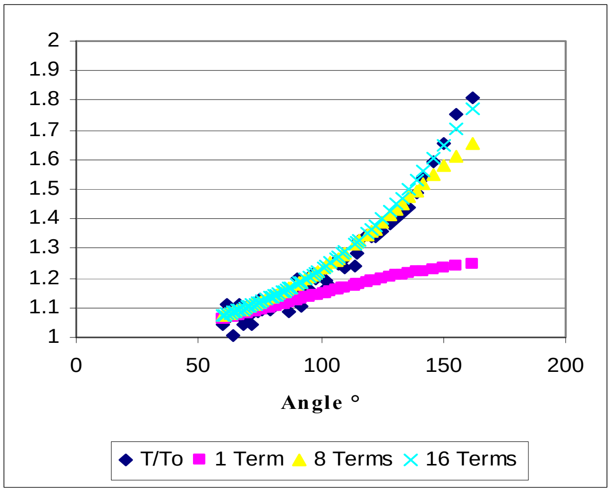

FIGURE 2. Measured values for T/T_0 and calculated values for the series obtained without the approximation *sin θ = θ*

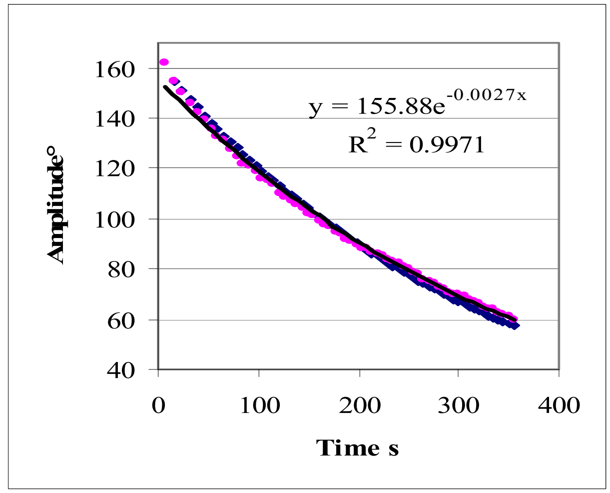

FIGURE. 3. Measured and fitted decaying amplitude of the oscillation. Experimental values as points and fitted equation the line.

III. FRICTION

We can make a numerical calculation to take into account the friction, either as a force proportional to the velocity or to the square of the velocity.

Equation (8) gives the torque from the Earth magnetic field B, for small amplitudes ($\sin\theta \approx \theta$) the period is $T = 2\pi\sqrt{I/MB}$.

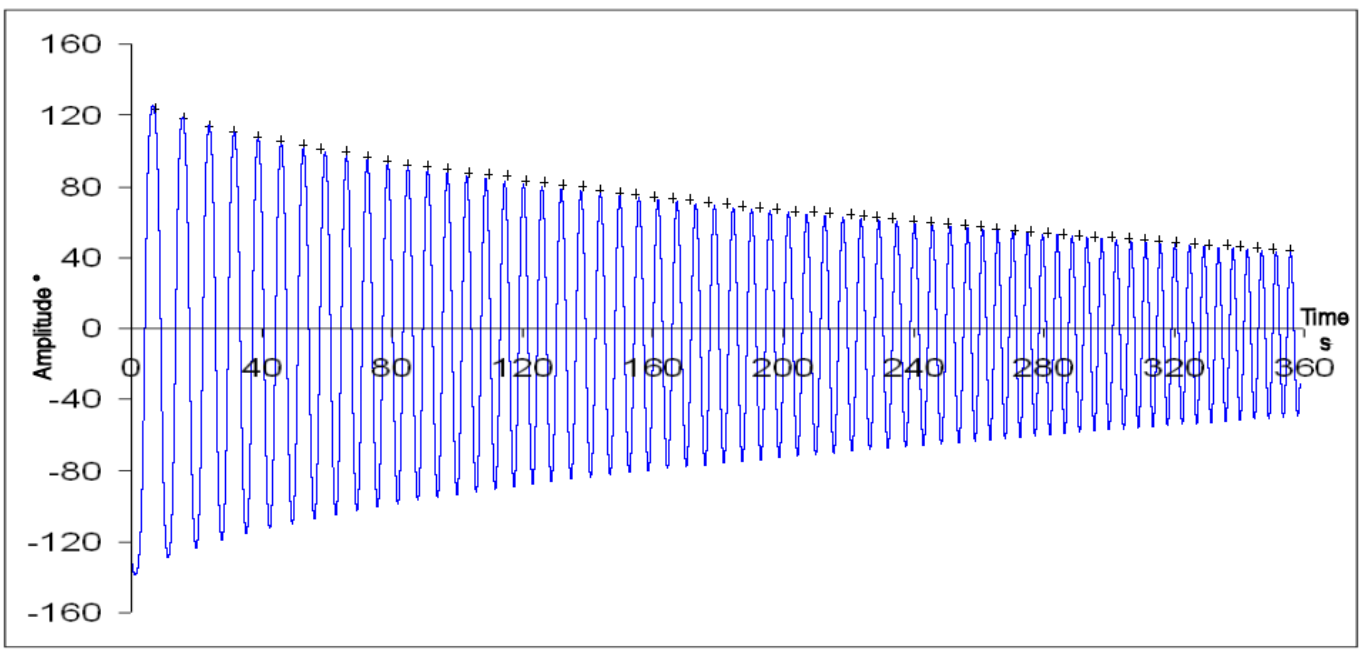

FIGURE 4. Calculated oscillations and maximum amplitude measured (+) for a friction force proportional to the angular velocity.

Assuming a friction proportional to the angular speed ω or to its square:

$$I(d^2\theta/dt^2) = -\lambda(d\theta/dt) - MB\sin\theta, \qquad (9)$$

where λ is the constant of proportionality for the friction force. Then:

$$d^2\theta/dt^2 = -(\lambda/I)*(d\theta/dt) - (MB/I)*\sin\theta = \alpha\,. \qquad (10)$$

To solve Eq. (10) numerically we need to fit values for the initial angle and angular velocity, and the values for λ/I and MB/I. The time increment Δt needs to be small compared with the period. The calculated solution should be indepen-

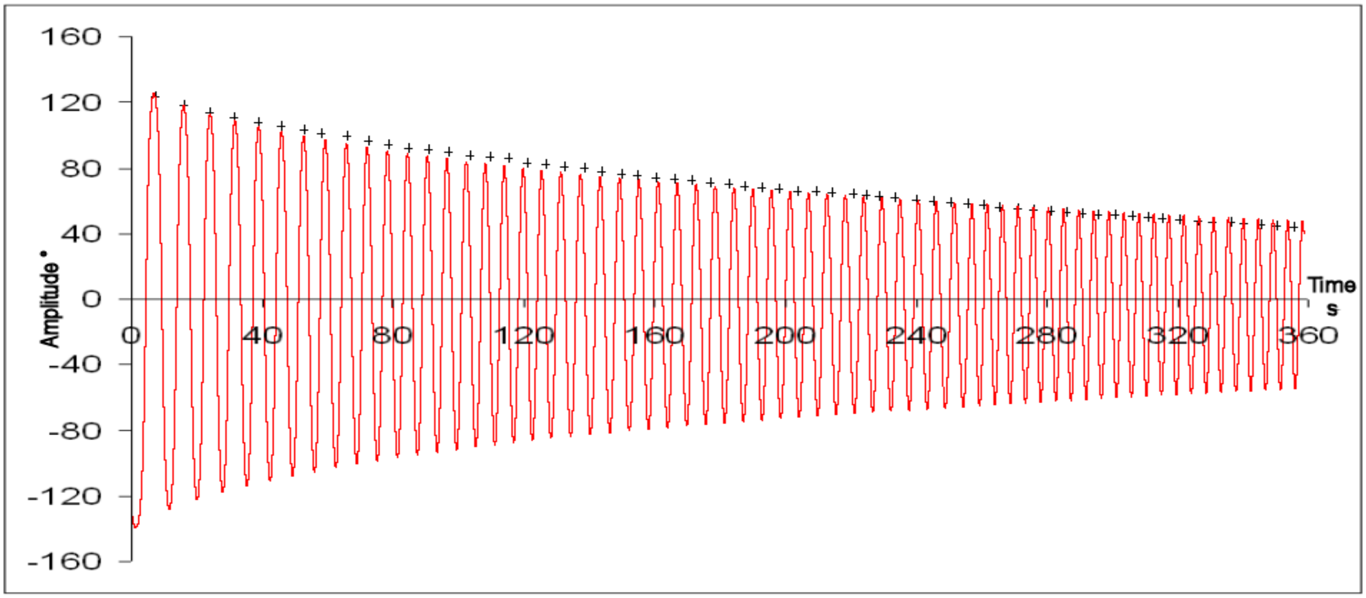

FIGURE 5. Calculated oscillations and maximum amplitude measured (+) for a friction force proportional to the square of the angular velocity.

dent of this value, reducing more than needed only increases the computer time.

The initial values are used to calculate the first value for α, then ($\omega = d\theta/dt$) the increment in angular velocity is

$$\Delta\omega = \alpha\Delta t, \quad (11)$$

$$\omega_f = \Delta\omega + \omega_i. \quad (12)$$

The increment in the angle θ is:

$$\Delta\theta = (\omega_i + \omega_f)*\Delta t/2, \quad (13)$$

$$\theta_f = \Delta\theta + \theta_i. \quad (14)$$

Using the calculated values, we can calculate the next value for the acceleration α making the same calculations for the next time interval. When calculating the friction force proportional to ω^2 is convenient to multiply ω by its magnitude so the product keeps the sign of ω, so the friction force is against the ω direction.

From the harmonic oscillator equation we obtain that $MB/I = 4*\pi^2/T_0^2$ where T_0 is the measured period for small amplitude oscillations.

Figures 4 and 5 show the agreement in amplitude and the period for the numerical calculation with an initial angle θ of -153° and initial angular velocity ω = -0.63 s^{-1}, λ/I = 0.0046 s^{-2} and MB/I = 1.97 s^{-2} (T_0 = 4.48 s) for friction proportional to angular speed. For a friction proportional to the square of ω, only changes λ/I = 0.0024 s^{-2} and MB/I = 1.93 s^{-2} (T_0 = 4.52 s). We try several Δt for the numerical calculations, each time reducing in half, until Δt = 0.0001s the amplitude for zero friction becomes constant and the period becomes independent of Δt.

Figure 4 shows agreement with the amplitude of the decaying oscillation and the period agreed up to 27 oscillations. Later the deviations are larger than the estimated 1 second uncertain.

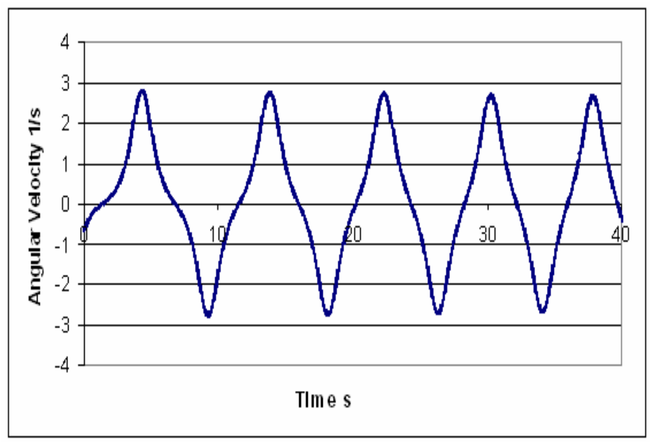

FIGURE 6. Initial cycles for the angular velocity ω showing a non-harmonic movement.

Figure 5 shows systematic deviation from the decaying amplitude and deviations also for the period. So the linear friction is a better approximation but it need some else to fit also the experimental change in period after 27 oscillations.

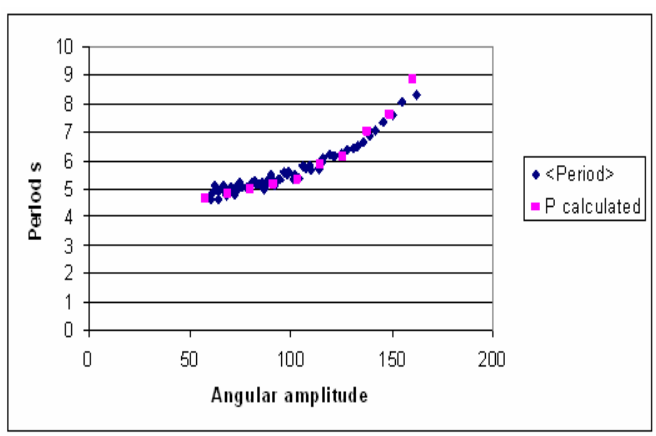

FIGURE 7. Moving average of three measured periods and calculated values for the period without friction.

The non-harmonic character of the movement is shown in Figure 6 with the angular velocity ω for a few initial cycles. Using the same numerical calculation with $\lambda/I = 0$ (without friction) we may calculate the period as a function of the amplitude.

Figure 7 shows the comparison between the moving average of three measured periods and the calculated values. Also shows that only for the maximum measured amplitude, the friction changes the period of the movement.

IV. CONCLUSIONS

The periodic movement of a horizontal magnet in the Earth magnetic field provides easy data for the amplitude and time measurements, which can be used to fit numerically the parameters of the equation related to the dipolar magnetic moment, inertia moment and friction. The friction changes the amplitude, but with slight effect in the period. For students at college level the data is a good example to show the power of numerical calculations and the need for moving averages. If a reader desires to exercise fitting parameters we can provide a program to fit a friction linear or quadratic for a time T, Δt, initial angle and angular velocity, period and friction coefficient.

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