Two forms of Wien's displacement law

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Abstract

The possible confusion of two form of Wien's law is clarified from the view points of physical significance and mathematics. In physics, because the spectral energy density distribution with frequency can't be simply converted to the distribution with wavelength by using $c=\lambda f$ the wavelength corresponding to the maximum spectral energy density can't be obtained by using $\lambda_{m=c/f_m}$ where f_m is the frequency corresponding to the maximum spectral energy density. In mathematics, because in the variables transformation from df to $d\lambda$by using $c=\lambda f$ there is an extra term $-c/\lambda^2$ the spectral energy density function is changed, so is its maximum position. We use a simple parabolic distribution function as an example to explain this problem more clearly.

Keywords: Wien's law, Planck function, peak frequency, peak wavelength.

Resumen

La posible confusión de dos formas de la ley de Wien es clarificada desde los puntos de vista del significado físico y matemático. En Física, debido a que la distribución espectral de la energía con la frecuencia no puede simplemente ser convertida a la distribución con longitud de onda usando $c=\lambda f$ la longitud de onda correspondiente al máximo de la densidad de energía espectral no puede ser obtenida por el uso de $\lambda_{m=c/f_m}$ donde f_m es la frecuencia correspondiente al máximo de densidad de energía espectral. En matemáticas, debido a que en las variables de transformación para pasar de df a $d\lambda$ por el uso de $c=\lambda f$ hay un término extra $-c/\lambda^2$ la función de densidad de energía espectral es cambiada, así también lo es su posición máxima. Usamos una simple distribución parabólica como ejemplo para explicar este problema más claramente.

Palabras clave: Ley de Wien, función de Planck, frecuencia pico, longitud de onda pico.

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There are two forms of Wien's displacement law that can be derived from Planck's equation [1, 2]. They are:

$$\lambda_m T = 2.89977685x10^{-3}\ m \bullet K\ , \tag{1}$$

$$f_m / T = 5.879x10^{10}\quad Hz/k\ . \tag{2}$$

Where λ_m and λ_f are wavelength and frequency corresponding to the maximum energy u_m of radiation of the black body, and T is the temperature of the black body. Suppose that we have known a black body's temperature, then λ_m and f_m can be obtained from Eqs. (1) and (2). For example, the Sun's surface temperature, $T=5778$K (see ref. [1]) and suppose that the sun can be regarded as a black body, then according to Eqs. (1) and (2), we get

$$\lambda_m = 5.015x10^{-7}m$$

and

$$f_m = 3.397x10^{14}\, Hz$$

However, if we apply $c=\lambda f$ and take c=3x10^8 m/s then from λ_m=5.015x10^{-7} m, we get f=5.982x10^{14} Hz, which is not the f_m obtained from Eq. (2). Why?

Let's take a look at how the Wien's law can be derived from Planck's function. For the black body radiation, the spectral energy density $u(f,T)$ (that is, the energy per unit volume per unit frequency), derived by Max Planck [3], is:

$$u\left(f,T\right) = \frac{8\pi h f^3}{c^3}\frac{1}{e^{\frac{hf}{kT}}-1}\ . \tag{3}$$

If the energy density is $U(T)$ which is energy per volume, then we have

$$U(T) = \int_0^\infty u\left(f,T\right)df\ . \tag{4}$$

Using the relation: $c=\lambda f$, we can change Eq. (3) to an expression for $u'(\lambda, T)$ in wavelength units by substituting f with c/λ and evaluating

$$u'(\lambda,T)=u(f,T)\left|\frac{df}{d\lambda}\right|. \qquad (5)$$

So the Eq. (4) is converted into:

$$U(T)=\int_0^\infty u'(f,T)\,d\lambda. \qquad (4')$$

Taking absolute value on $df/d\lambda$ is simply because the spectral energy density should be positive in physics. In mathematics, integration from low frequency to high frequency corresponds to the integration from long wavelength to short wavelength. Therefore, if we still integrate from short wavelength to long wavelength, we need to change the sign. From Eq. (5) we have

$$u'(\lambda,T)=\frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}. \qquad (6)$$

To find the f_m which corresponds to the peak of $u(f, T)$ for Eq. (3), we should let

$$\frac{du(f,T)}{df}=0, \qquad (7)$$

which yields Eq. (2) [4]. On the other hand, to find λ_m which corresponds to the peak of $u'(\lambda, T)$ for Eq. (6), we should let

$$\frac{du'(\lambda,T)}{d\lambda}=0, \qquad (8)$$

which yields Eq. (1). The validity of two forms of Wien's law is not only justified by the derivation from Planck's function, but also by the fact that they fit experimental data well [5].

Mathematically we may understand the two forms of Wien's law easily. From Eqs. (4) and (4'), we can see that because we need $U(T)=\int_0^\infty u'(\lambda,T)\,d\lambda=\int_0^\infty u(f,T)df$ rather than $u(f, T) = u'(\lambda, T)$ we cannot simply replace f with c/λ [6]. Here we give another example for pedagogical purpose. Suppose that the distribution of z as a function of x is in the parabolic form:

$$z(x)=-x^2+x+1, \qquad (9)$$

And $y=1/x$. Suppose that the areas under the curves of $z(x)$ and $z(y)$ are, just like the Eqs. (4) and (4'), the same. That is: $\int_{x_1}^{x_2} z(x)dx=\int_{y_1}^{y} z'(y)dy$, where $y_1=1/x_2$ and $y_2=1/x_1$. So the distribution of z' as a function of y is:

$$z'(y)=z(y)\left|\frac{dx}{dy}\right|.$$

Therefore

$$z'(y)=-\frac{1}{y^4}+\frac{1}{y^3}+\frac{1}{y^2}. \qquad (10)$$

The peak of the $z(x)$ for Eq. (7) is at (0.5, 1.25) and there are 3 places that $dz(y)/dy=0$: (-2.35, 0.0712), maximum; (0,-∞), minimum; and (0.851,1.10), maximum; as shown in the figure. We can see that one would get $y=2$ if he uses $y=1/x$ directly to get the maximum/minimum place; that corresponds to none of those $z(y)$ real maximum/minimum positions. Therefore, for the two forms of Planck's function, it is reasonable to get λ_m and f_m that $c\neq \lambda_m f_m$ Actually, product of Eqs. (1) and (2) gives

$$\lambda_m f_m=1.705x10^8 m/s=c/1.760, \qquad (11)$$

where c is the speed of light. In other words, the real λ_m is always smaller than the value λ if one gets f_m from Eq. (2) and then applies $c=\lambda f$ to get λ. Equation (11) agrees with the conclusions of Ref. [5].

Therefore, when one wants to know the λ_m, he needs to use Eq. (1) to get it; when one wants to know f_m, he needs to use Eq. (2) or (11) to get it. Using $c=\lambda f$ can give him the other quantity but it is not the one corresponding to the maximum spectral energy density, although the $c=\lambda f$ is still right. However, in practice, what happens if one uses a spectrometer to measure the spectrum of the light of the Sun? Does he get the u_m position at λ_m or f_m? It really depends on what kind of spectrometer he uses. If he uses the spectrometer that measures $u_m(\lambda, T)$, then he gets λ_m; if he uses the spectrometer that measures $u_m(f, T)$, then he gets f_m. In either case however, he should use Eq. (11), rather than $c=\lambda f$, to get one quantity from another quantity.

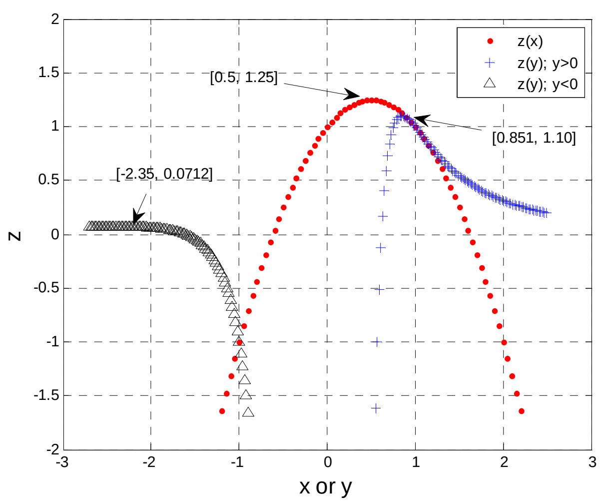

FIGURE 1. Distribution function z as a function of x or y from Eqs. (9) and (10) respectively.

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