

# Classical velocity transformations



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## Abstract

The Galilean transformation, relating the velocities of a particle observed in two different frames of reference, must be modified if one frame is rotating relative to the other. Introductory courses typically do not mention this fact, inviting subsequent confusion about the kinematics and dynamics of combined translational-rotational motion.

**Keywords:** Galilean transformation, relative velocity, translations, rotations.

## Resumen

La transformación de Galileo, sobre la velocidad de una partícula observada en dos diferentes marcos de referencia, debe ser modificado si se está girando con respecto al otro. Cursos de introducción general, no mencionan este hecho, invitando a la confusión posterior acerca de la cinemática y la dinámica de la combinación de movimiento de traslación, rotación.

**Palabras clave:** Transformación de Galileo, velocidad relativa, traslaciones, rotaciones.

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Suppose that a particle P is observed by observers in two different reference frames, A and B, as sketched in Fig. 1. One sees from the figure that

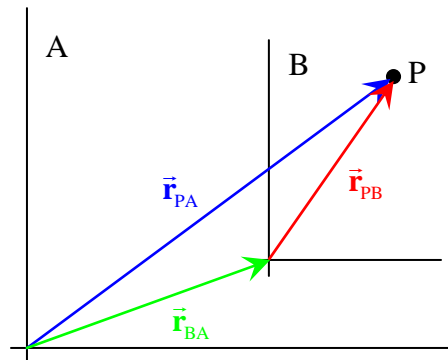
$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA} \quad (1)$$

in complete generality. Here  $\vec{r}_{PA}$  is the position of the particle relative to the origin of the A coordinate system,  $\vec{r}_{PB}$  is the position of P relative to the origin of frame B, and  $\vec{r}_{BA}$  is the location of the origin of coordinate system B as measured in frame A. It would seem that if one took the time derivative of Eq. (1) one would establish that

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA} \quad (2)$$

must also be true with complete generality (assuming all speeds are small compared to that of light), where  $\vec{v}_{PA}$  is the velocity of the particle as measured in frame A,  $\vec{v}_{PB}$  is the velocity of P measured by observer B, and  $\vec{v}_{BA}$  is the velocity of the origin of coordinate system B from the point of view of reference frame A. But that conclusion is wrong! Consider the following counter-example. Let observer B be on a rotating carousel platform and observer A be on the ground. Define both coordinate systems to initially coincide, with the coordinate origin located on the axis of rotation of the platform. Then both  $\vec{r}_{BA}$  and  $\vec{v}_{BA}$  are always zero, because the origins of frame A (attached to the ground) and frame B (fixed on the carousel) never move. Now let P be a

dot painted on the surface of the rotating platform near its circumference.



**FIGURE 1.** A particle P whose position is measured relative to two different coordinate systems, A and B.

The dot is always at rest from the point of view of observer B, so that  $\vec{v}_{PB} = 0$ . But from the vantage point of observer A, the dot is always moving and consequently  $\vec{v}_{PA}$  is never zero, in contrast to what Eq. (2) predicts!

The limited validity of Eq. (2) stems from the fact that

$$\frac{d}{dt}(\vec{r}_{PA} - \vec{r}_{PB}) \neq \vec{v}_{PA} - \vec{v}_{PB} \quad (3)$$

in general, because the time derivative of a vector is frame dependent [1]. The right-hand side of Eq. (3) is the time

derivative of  $\vec{r}_{PA}$  evaluated in frame A minus the time derivative of  $\vec{r}_{PB}$  evaluated in frame B. On the other hand, using Eq. (1) the left-hand side of Eq. (3) reduces to a single time derivative of  $\vec{r}_{BA}$  measured in frame A. One of these three time derivatives is thus computed in a different reference frame than than the other two and so Eq. (2) does not always hold true.

Thinking perhaps of such examples, a few introductory textbooks qualify Eq. (2). For instance, Knight [2] requires the two frames to be co-inertial. But in addition to the fact that it is awkward to discuss the concept of an inertial frame in the kinematics chapters prior to subsequently introducing Newton's laws, this requirement is overly restrictive. Equation (2) is valid if one frame has a purely translational acceleration relative to the other frame. For example, it holds for a projectile P observed both from a cart A accelerating down an inclined plane and from a second cart B traveling along level ground. In general Eq. (2) is valid if and only if frame B is not rotating relative to frame A and therefore that equation should more properly be called the "translational Galilean transformation." One can later include rotations to obtain the "generalized Galilean transformation,"

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA} + \vec{\omega}_{BA} \times \vec{r}_{PB}. \quad (4)$$

Here  $\vec{\omega}_{BA}$  is the angular velocity of frame B about its origin as observed in frame A. Equation (4) can be derived by combining Eq. (2) from translational kinematics with  $\vec{v} = \vec{\omega} \times \vec{r}$  for rigid-body rotations. (Alternatively in a higher level course, the cross-product term can be obtained by formally establishing the transformation relation for an arbitrary time derivative between frames of reference [1].) There are several advantages to introducing this generalized

*Three Important Taylor Series for Introductory Physics* version of the Galilean transformation in the introductory course for physics majors. The result greases the way to present the Coriolis and centrifugal forces in the intermediate mechanics course. It also helps explain the decomposition of angular momentum into orbital and spin terms (say for the earth), as well as of kinetic energy into translational and rotational terms (such as for a rolling wheel). It furthers the parallel development of concepts in the chapters treating translational and rotational kinematics. Finally it is relevant to discussions of how other quantities such as work and impulse transform between reference frames [3, 4]. However, even if instructors of the introductory course choose to omit discussion of the generalized version of the Galilean transformation, they still should be careful not to imply that the derivation of Eq. (2) from Eq. (1) is either simpler or more general than it actually is!

## REFERENCES

- [1] Thornton, S. T. and Marion, J. B., *Classical Dynamics of Particles and Systems*, 5<sup>th</sup> ed. (Thomson Brooks-Cole, Belmont CA, 2004), Sec. 10.2.
- [2] Knight, R. D., *Physics for Scientists and Engineers*, 2<sup>nd</sup> ed., (Pearson Addison-Wesley, San Francisco CA, 2008), Sec. 4.4.
- [3] Mallinckrodt, A. J. and Leff, H. S., *All about work*, Am. J. Phys. **60**, 356–365 (1992).
- [4] <<http://usna.edu/Users/physics/mungan/Scholarship/WorkTransformation.pdf>> Visited September 8, 2009.