The resonance of the Wilberforce pendulum and the period of beats

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Abstract

The resonance of the Wilberforce pendulum is defined as the state of the maximum period of beats. The plausible assertion that resonance is characterized by the equal values of both the frequencies of longitudinal and torsion vibrations is proven. Although a coupling constant between longitudinal and torsion vibrations determines the frequencies of normal modes it plays no role in the definition of the resonance state.

Keywords: Wilberforce pendulum, resonance, beats.

Resumen

La resonancia del péndulo de Wilberforce se define como el estado de la duración máxima de ritmos. La afirmación plausible de que la resonancia se caracteriza por los valores iguales tanto de las frecuencias longitudinales como de las vibraciones de torsión se demuestra. Aunque una constante de acoplamiento entre las vibraciones longitudinales y las de torsión determina las frecuencias de los modos normales, no desempeña ningún papel en la definición del estado de resonancia.

**Palabras clave:** Péndulo de Wilberforce, resonancia, ritmos.

In 1895 Wilberforce invented a pendulum [1] that shows a transfer of energy from the longitudinal to the torsion oscillations of a mass attached to a long helical spring. This effect has attracted the attention of many authors [2, 3, 4, 5, 6]. The common feature of the approaches of these authors is the assumption that the Wilberforce pendulum is in a state of resonance when the frequencies of the uncoupled oscillations are equal. However these oscillations are coupled and normal frequencies are functions of the coupling constant.

In this paper we put the question of resonance on firm ground by stating that the state of resonance of the Wilberforce pendulum is the state of the maximum period of beats. In this way one can experimentally check the resonance state of the Wilberforce pendulum.

The standard model of the Wilberforce pendulum [2, 3, 4, 5, 6] assumes a cylindrical and symmetric body of mass suspended on a massless helical spring. Due to the constant length of the spring wire the longitudinal extension or compression of the spring induces small changes in the radius of the spring. These changes induce the torsion of the spring around its axis. This effect is schematically shown in Fig. 1. We assign the coordinates z and φ to the longitudinal and torsion deformation of the spring (see Fig. 1). The origin of the coordinate system is the equilibrium position of the body. Then the body weight G plays no role in the dynamics of the system.

The corresponding Lagrangian function [4] is

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\varphi}^2 - \frac{1}{2}kz^2 - \frac{1}{2}D[\varphi - \varphi(z)]^2. \tag{1}$$
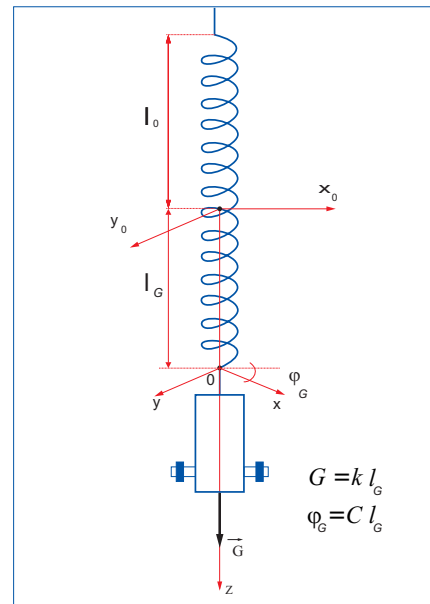
FIGURE 1. The helical spring of the unstretched length I_0 and corresponding coordinates. G is the body weight, I_G and φ_G are, respectively, the extension and torsion angle of the spring due to the weight of the suspended body. Origin 0 of the x, y, z coordinate system is related to the equilibrium of the system.

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We measure the torsion angle from the equilibrium position. Then

$$\varphi(z) = Cz, \qquad (2)$$

describes the torsion of the spring due to the longitudinal displacement z. The body mass is m, I is its moment of inertia, k is the static spring longitudinal constant and D is the torsion constant. Putting $k' = k + DC^2$, Eq. (1) becomes

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\varphi}^2 - \frac{1}{2}k'z^2 + DC\varphi z - \frac{1}{2}D\varphi^2. \qquad (3)$$

The equations of motions are

$$m\ddot{z} = -k'z + DC\varphi, \qquad (4)$$

$$I\ddot{\varphi} = DCz - D\varphi. \qquad (5)$$

Assuming the solution in the form of

$$z = Ae^{i\omega t + \theta}, \qquad (6)$$

$$\varphi = Be^{i\omega t + \theta} \qquad (7)$$

we get a system of two homogenous linear equations for A and B,

$$(m\omega^2 - k')A + DCB = 0, \qquad (8a)$$

$$DCA + (I\omega^2 - D)B = 0. \qquad (8b)$$

This system has a nontrivial solution if the determinant of the system vanishes, *i.e.* if

$$\begin{vmatrix} k' - m\omega^2 & -DC \\ -DC & D - I\omega^2 \end{vmatrix} = 0. \qquad (9)$$

The zeroes of the corresponding biquadrate equation

$$mI\omega^4 - (k'I + mD)\omega^2 + kD = 0, \qquad (10)$$

determine the frequencies of the normal modes,

$$\omega_{1,2}^2 = \frac{k'I + mD \pm \sqrt{(kI - mD)^2 + 2(kI + mD)DC^2I + D^2I^2C^4}}{2mI}. \qquad (11)$$

When the coupling between the longitudinal and torsion motion is absent ($C=0$), these frequencies are equal to the frequencies of the uncoupled longitudinal and torsion modes $\omega_{0_1}^2 = k/m$, $\omega_{0_2}^2 = D/I$, respectively.
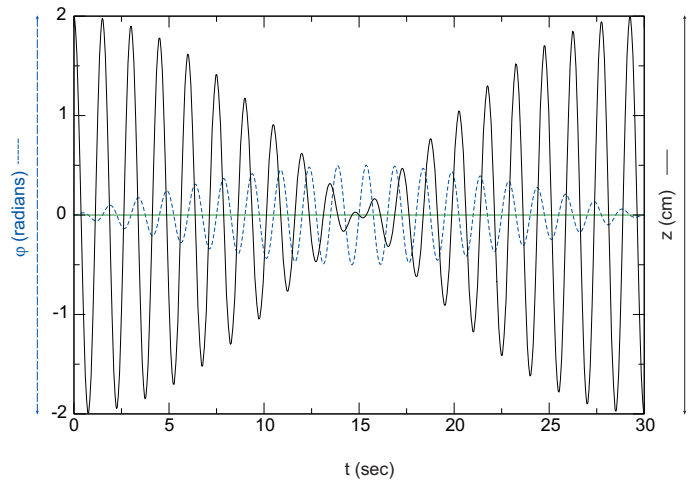
The general solutions for the coordinates are

$$z = A_1 e^{i\omega_1 t + \theta_1} + A_2 e^{i\omega_2 t + \theta_2}, \qquad (12)$$

$$\varphi = B_1 e^{i\omega_1 t + \theta_1} + B_2 e^{i\omega_2 t + \theta_2}. \qquad (13)$$

The constants A_i and B_i are coupled by Eqs. (8). We need four conditions, in addition to Eqs. (8), to find out the time dependences of the coordinates. These are the initial conditions for the generalized coordinates and corresponding velocities. Choosing $z(0) = A$ and $\varphi(0) = \dot{\varphi}(0) = \dot{z}(0) = 0$ the time dependence of coordinates read,

$$z = A\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right), \qquad (14)$$

$$\varphi = B\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\sin\left(\frac{\omega_1 - \omega_2}{2}t\right). \qquad (15)$$

FIGURE 2. The coordinates of Wilberforce pendulum as functions of time.

The solutions are shown in Fig. 2. The frequency of beats is equal to $f_b = (\omega_1 - \omega_2)/2\pi$.

Turning disk-like screw nuts (see Fig. 1) down the bolt we change the moment of inertia of the body and subsequently the frequencies of the uncoupled torsion vibration and beats, respectively. Experimentally measurable quantities are the normal frequencies or their algebraic sums, $\omega_1 - \omega_2$ and the frequency of beats $f_b(I)$. As we stated at the beginning of this paper we have chosen the latter to define the resonance of the Wilberforce pendulum. During experimentation the moment of inertia was changed until the largest possible period of beats is reached.

For the sake of simplicity we are looking for the minimum of $(\omega_1 - \omega_2)^2$ rather than of $\omega_1 - \omega_2$. Both functions achieve their minima for the same value of the moment of inertia.

Using Eq. (11) after lengthy but otherwise straightforward calculations we get

$$(\omega_1 - \omega_2)^2 = \frac{(k + DC^2)I + mD}{mI} - 2\sqrt{\frac{kD}{mI}}. \qquad (16)$$

The first derivation of this function

$$\frac{d(\omega_1 - \omega_2)^2}{dI} = -\frac{D}{I^2} + \frac{1}{I}\sqrt{\frac{kD}{mI}}, \qquad (17)$$

vanishes for

$$\frac{k}{m} = \frac{D}{I} \quad or \quad \omega_{0_1} = \omega_{0_2}. \qquad (18)$$

Resonance occurs if the frequencies of uncoupled vibrations are equal. An interesting description of the Wilberforce pendulum based on continuum mechanics is provided by Köpf [5]. He has shown that the resonance state of the Wilberforce pendulum does not depend on length or diameter of the wire nor on the pitch or number of turns of the spring. Here we have shown another peculiarity of this pendulum. The coupling constant between longitudinal and torsion vibrations does not enter into a condition for resonance although the resonance condition is defined in

terms of the period of beats, which in turn is a function of the coupling constant.

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