A ball rolling down a rotating board

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Abstract

This article describes an easily constructed apparatus for an experiment on a rolling ball a rotating board. From the experimental and theoretical results, we can find that the ball leaves the board at the instant both the ball and the board are released from hand. Then, the ball can't roll down the board until the ball touches the board again.

Keywords: Rolling ball, Rotating board, Non-linear differential equation.

Resumen

En este artículo se describe un aparato de fácil construcción para un experimento con una bola que rueda en un plano inclinado giratorio. De los resultados experimentales y teóricos, podemos encontrar que el balón sale del plano en el instante en que la bola y el plano inclinado son liberados de la mano. Entonces, la bola no puede rodar por la tabla hasta que la bola toca la tabla de nuevo.

Palabras clave: Bola rodante, rotación de plano, ecuación diferencial no lineal.

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INTRODUCTION

A ball starts from rest and rolls down an inclined board. This example is printed in many textbooks of physics. If the board rotates freely, how far has the ball rolled down the board when the board becomes horizontal?

PROBLEM 1

Let us consider a following problem. Two boards are fixed together by a hinge at one end so as to allow them to fold together. One board is laid flat on the table and the other board (mass = M, length = ℓ) is inclined, as shown in Fig.1, with a ball (mass = m, radius = r) placed on the other end of the board. The board forms an angle θ_0 with the vertical. When both the ball and the board are released from hand, the board pivots freely on a horizontal axis O' through the hinge, and the ball starts from rest and rolls down the board without slipping. When the board becomes horizontal, how far has the ball rolled down the board? If the ball leaves the board before the board reaches the bottom, what are those conditions? Assume that friction about the horizontal axis O' and air resistance are negligible.

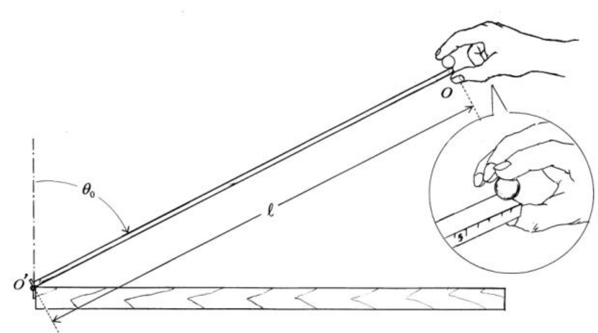

FIGURE 1. A ball of radius *r* rolls without slipping, and a board of length ℓ is freely rotated about a horizontal axis O'.

DISCUSSION 1

Using an apparatus shown in Fig.1, experiments were carried out. At the instant both the ball and the board are released from hand, the ball appears to leave the board. Hence, we consider the normal reaction N of the board shown in Fig. 2.

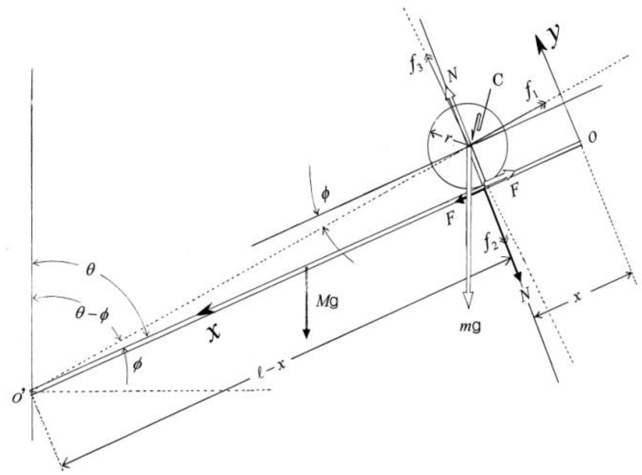

FIGURE 2. A ball is acted on by the six forces $m\mathrm{g}$, F, N, f_1, f_2 and f_3. The forces f_1, f_2 and f_3 are the inertial ones, and shown by the double arrows. A board is acted on by the three forces $M\mathrm{g}$, N, and a certain force on O'.

The forces acting on the ball will be the force $m\mathrm{g}$ of gravity, the force of friction F and N, where g is the acceleration of gravity. Let O be the origin of the coordinate system (x, y). The direction of $x-$ axis is parallel to the inclined board along downward. The direction of $y-$axis is normal to the board. The ball is on the $x-$ axis at the point x . By Newton's Second Law of Angular Motion, the moment of inertia times the clockwise angular acceleration will equal the clockwise torque about O' of the board at an arbitrary time:

$$\frac{1}{3}M\ell^2\ddot{\theta} = \frac{1}{2}M\mathrm{g}\ell\sin\theta + (\ell - x)N, \qquad (1)$$

where $\frac{1}{3}M\ell^2$ is the moment of inertia about O' of the board.

On the other hand, there are three inertial forces acting on the ball:

Centrifugal force $\qquad f_1 = m\sqrt{(\ell - x)^2 + r^2}\ \dot{\theta}^2.$

Coriolis force $\qquad f_2 = 2m\dot{x}\dot{\theta}.$

Angular acceleration force $\quad f_3 = m\sqrt{(\ell - x)^2 + r^2}\ \ddot{\theta}.$

Including these inertial forces, the forces normal to the board will balance, since there is no acceleration of the center of the ball in $y-$axis direction.

Therefore,

$$N + m\sqrt{(\ell - x)^2 + r^2}\ \ddot{\theta}\cos\phi + m\sqrt{(\ell - x)^2 + r^2}\dot{\theta}^2\sin\phi$$
$$-2m\dot{x}\dot{\theta} - m\mathrm{g}\sin\theta = 0. \qquad (2)$$

Where ϕ is the angle CO'O, and point C is the center of the ball.

Using following equations, $\sin\ \phi = \frac{r}{\sqrt{(\ell - x)^2 + r^2}}$ and $\cos\ \phi = \frac{\ell - x}{\sqrt{(\ell - x)^2 + r^2}}$, we can rewrite Eq. (2) into

$$N + m(\ell - x)\ddot{\theta} + mr\dot{\theta}^2 - 2m\dot{x}\dot{\theta} - m\mathrm{g}\sin\theta = 0. \qquad (2a)$$

For time $t = 0$ where both ball and board are initially at rest, $x\big|_{t=0} = 0,\ \theta\big|_{t=0} = \theta_0,\ \dot{x}\big|_{t=0} = 0,\ \dot{\theta}\big|_{t=0} = 0$, Eqs. (1) and (2a) become

$$\frac{1}{3}M\ell^2\ddot{\theta}\Big|_{t=0} = \frac{1}{2}M\mathrm{g}\ell\sin\theta_0 + \ell N\Big|_{t=0}. \qquad (1b)$$

And

$$N\Big|_{t=0} + m\ell\ddot{\theta}\Big|_{t=0} - m\mathrm{g}\sin\theta_0 = 0. \qquad (2b)$$

Combining these two equations, we obtain

$$N\Big|_{t=0} = -\frac{mM\mathrm{g}\sin\theta_0}{6m + 2M}. \qquad (3)$$

Eq. (3) shows that $N\big|_{t=0}$ has a negative value for $0 < \theta_0 \le \frac{\pi}{2}$. Then, at the instant both the ball and the board are released from hand, the ball leaves the board, regardless of initial angle θ_0. Thus, the ball falls freely and the board rotates as expressed in Eq. (1) with $N = 0$. Then, we have the following results:

$$\textbf{Ball}\ \ y = \frac{1}{2}\mathrm{g}t^2. \qquad (4)$$

$$\textbf{Board}\ \ \frac{1}{3}M\ell^2\ddot{\theta} = \frac{1}{2}M\mathrm{g}\ell\sin\theta. \qquad (5)$$

The ball appears to collide with the board before the board becomes horizontal under certain conditions from experimental observation.

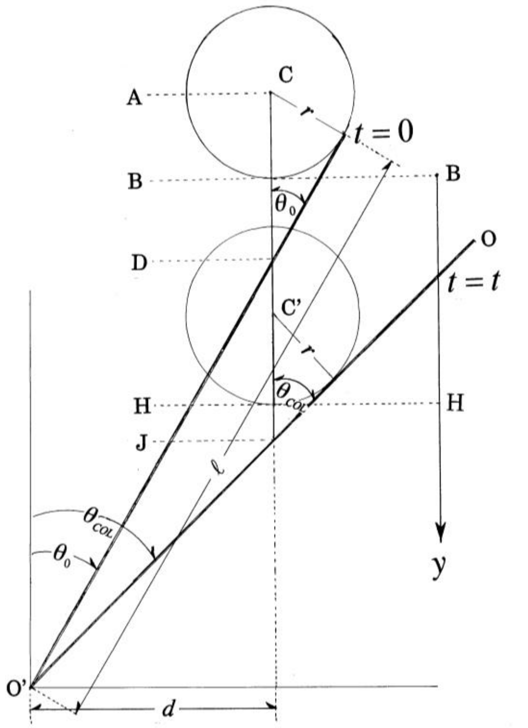

FIGURE 3. The ball falls freely from B to H during the time t.

As shown in Fig. 3, when the ball collides with the board, the ball falls a distance of $\overline{BH} = \frac{1}{2}gt^2$ in the time t. Let us assume that the board forms an angle θ_{COL} with the vertical at the time $t = t$. Then, we can write

$$\overline{BH} = \frac{1}{2}gt^2,$$

$$\overline{BD} + \overline{DJ} - \overline{HJ} = \frac{1}{2}gt^2,$$

$$r(\frac{1}{\sin\theta_0} - 1) + d(\frac{1}{\tan\theta_0} - \frac{1}{\tan\theta_{COL}}) - \frac{1}{r}(\frac{1}{\sin\theta_{COL}} - 1) = \frac{1}{2}gt^2,$$

$$r(\frac{1}{\sin\theta_0} - 1) + (\ell\sin\theta_0 - r\cos\theta_0)(\frac{1}{\tan\theta_0} - \frac{1}{\tan\theta_{COL}}),$$

$$-\frac{1}{r}(\frac{1}{\sin\theta_{COL}} - 1) = \frac{1}{2}gt^2,$$

$$r(\frac{1}{\sin\theta_0} - \frac{1}{\sin\theta_{COL}}) + (\ell\sin\theta_0 - r\cos\theta_0)(\frac{1}{\tan\theta_0} - \frac{1}{\tan\theta_{COL}}),$$

$$= \frac{1}{2}gt^2. \quad (6)$$

Using the computer "software "*Mathematica* [1]", we can calculate numerical values of the angle θ_{COL} for different θ_0 from Eqs. (5) and (6). For example, for $\theta_0 = \frac{6\pi}{32}[rad]$ the ball collides with the board forming $\theta_{COL} = \frac{11\pi}{32}[rad]$ at the time t=0.178[sec]. For $\theta_0 = \frac{6\pi}{32}[rad]$, the positions of the ball and the board at the time $t = 0$, 0.118, and 0.178 [sec] are shown in Fig.4.

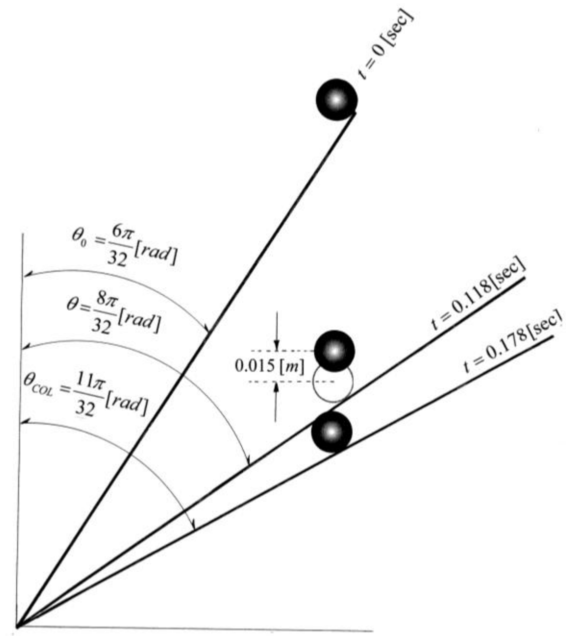

FIGURE 4. Illustrating the position of the ball and the board at the time *t=0*, 0.118, and 0.178 [sec].

Fig.5 shows experimentally observed collision between falling ball and rotating board recorded by using a high-speed digital video camera at 500 frames per second rate. In initial state (Fig. 5(a)), the ball and the board are the largest length, 0.015 m, apart in vertical direction at the time $t = 0.118$ [s]. The time is estimated from the number of frames. In Fig. 5(b), the ball collides with the board at the time $t = 0.178$ [s]. The result is in good agreement with the calculated data shown in Fig.4.

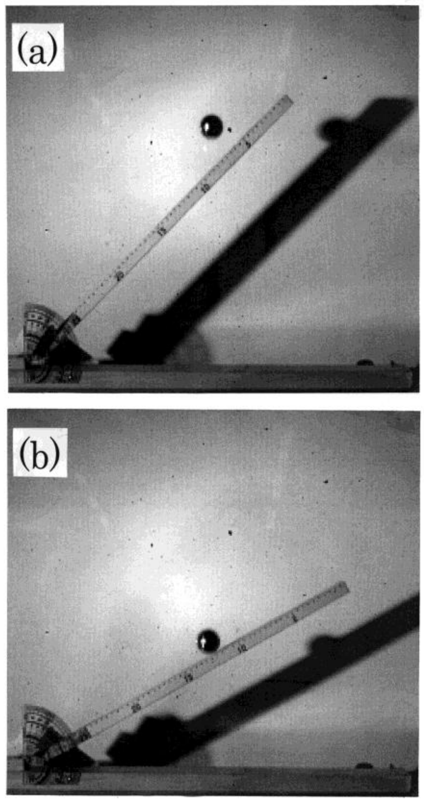

FIGURE 5. (a) The ball and the board are 0.015 m apart in vertical direction at the time t=0.118[s]. (b) The ball collides with the board at the time t=0.178 [s].

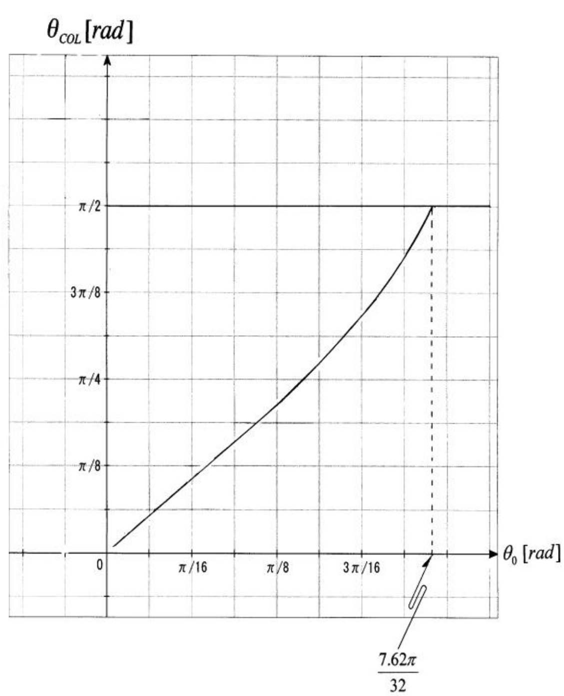

FIGURE 6. θ_{COL} as a function of θ_0. The solid line is calculated.

CONCLUSION 1

Fig.6 shows how the angle θ_{COL} depends on θ_0. From Fig.6 we see that the ball collides with the board before the board becomes horizontal for $0 < \theta_0 \le \frac{7.62\pi}{32}$. Hence, the ball cannot collide with the board before the board becomes horizontal for $\theta_0 > \frac{7.62\pi}{32}$. This problem is similar to the problem of "Falling Chimney [2, 3]".

Therefore, we made an apparatus shown in Fig.7 so that the ball is to remain in contact with the board at all times. In this apparatus, a ball with hole and a rod are used instead of the ball and the board.

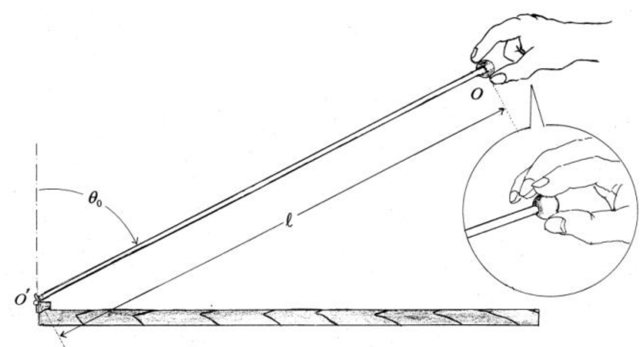

FIGURE 7. A ball slides along a rotating rod.

PROBLEM 2

When both the ball (mass = m) and the rod (mass = M, length = ℓ) are released from hand, the rod pivots freely on a horizontal axis O' through the hinge, and the ball starts from rest and slides along the rod. When the rod becomes horizontal, how far has the ball slid down the rod? Assume that friction and air resistance are negligible.

DISCUSSION 2

At an arbitrary time, the total kinetic energy T is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\ell - x)^2\dot{\theta}^2 + \frac{1}{2}\cdot\frac{1}{3}M\ell^2\dot{\theta}^2,$$

where $\frac{1}{2}m\dot{x}^2$ and $\frac{1}{2}m(\ell - x)^2\dot{\theta}^2$ are translational and rotational kinetic energy of the ball, and $\frac{1}{2}\cdot\frac{1}{3}M\ell^2\dot{\theta}^2$ is rotational kinetic energy of the rod. The potential energy of the ball and the rod respectively are $(\ell\text{-}x)m\mathrm{g}\cos\theta$, and $\frac{1}{2}M\mathrm{g}\ell\cos\theta$. At the bottom we have taken the potential energy of both the ball and the rod to be zero.

Then, the total potential energy U is

$$U = (\ell\text{-}x)m\mathrm{g}\cos\theta + \frac{1}{2}M\mathrm{g}\ell\cos\theta.$$

Therefore, Lagrange function L is

$$L = T - U,$$

$$= \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}m(\ell-\dot{x})^2\dot{\theta}^2 + \frac{1}{6}M\ell^2\dot{\theta}^2,$$

$$-(\ell-x)m\mathrm{g}\cos\theta - \frac{1}{2}M\mathrm{g}\ell\cos\theta. \qquad (7)$$

The motion of the system is given by the two Lagrange equations. The first equation of motion is written in the form.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Then,

$$\frac{d}{dt}\left[m\dot{x}\right] - m(\ell-x)(-1)\dot{\theta}^2 - m\mathrm{g}\cos\theta = 0,$$

$$m\ddot{x} + m(\ell-x)\dot{\theta}^2 - m\mathrm{g}\cos\theta = 0. \qquad (8)$$

The second equation of motion is written in the form.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0.$$

Then,

$$\frac{d}{dt}\left[m(\ell-x)^2\dot{\theta} + \frac{1}{3}M\ell^2\dot{\theta}\right]$$

$$-\frac{1}{2}M\mathrm{g}\ell\sin\theta - (\ell-x)m\mathrm{g}\sin\theta = 0,$$

$$-2m(\ell-x)\dot{x}\dot{\theta} + m(\ell-x)^2\ddot{\theta} + \frac{1}{3}M\ell^2\ddot{\theta} - \frac{1}{2}M\mathrm{g}\ell\sin\theta$$

$$-(\ell-x)m\mathrm{g}\sin\theta = 0. \qquad (9)$$

The initial conditions are $\theta\big|_{t=0} = \theta_0$, $x\big|_{t=0} = 0$, $\dot{\theta}\big|_{t=0} = 0$, and $\dot{x}\big|_{t=0} = 0$. The constant values are $m = 0.300[kg]$, $M = 0.150[kg]$, $\ell = 0.300[m]$ and $\mathrm{g} = 9.80[m/s^2]$. When the rod becomes horizontal, $\theta = \frac{\pi}{2}$ and $x = X$. Eqs. (8) and (9) are non-linear differential equations (of order 2), and these can be rarely solved. Using "*Mathematica*", we can calculate numerical values of the distance X for different initial angle θ_0.

CONCLUSION 2

Fig. 8 shows how the X depends on θ_0. The solid curve is calculated curve and closed circles (•) are experimental points. The experimental results were in good agreement with the calculated values although friction and air resistance are neglected.

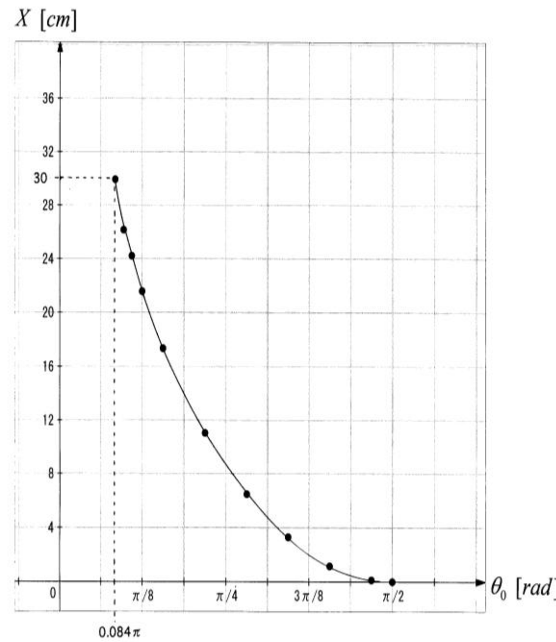

FIGURE 8. X as a function of θ_0. The solid curve is calculated, and closed circles (•) are experimental points.

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