Eigen spectra for Woods-Saxon plus Rosen-Morse potential

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Abstract
We investigate the solution of Woods-Saxon plus Rosen-Morse potential by using Nikiforov-Uvarov method. The eigenvalues and eigenfunctions of this potential are obtained. The results include the energy spectrum of the Woods-Saxon potential and Rosen-Morse potential. The PT and non PT -symmetric solutions for this potential are also presented.

Keywords: Schrödinger equation, Woods-Saxon potential, Rosen-Morse potential.

Resumen
Investigamos la solución de Woods-Saxon más el potencial de Rosen-Morse, utilizando el método Nikiforov-Uvarov. Se obtuvieron los valores propios y las funciones propias de este potencial. Los resultados incluyen el espectro de energía del potencial de Woods-Saxon y el potencial de Rosen-Morse. También se presentan las soluciones PT y no-PT simétricas para este potencial.

Palabras clave: Ecuación de Schrödinger, potencial de Woods-Saxon, potencial de Rosen-Morse.

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I. INTRODUCTION

It is well known that the exact solutions of the Schrödinger equation for some physical potentials are very important since they contain all the necessary information for the quantum system under consideration. These potentials are the harmonic oscillator potential [1], Kratzer potential [2, 3], Eckat potential [4], Pöschl–Teller potential [5, 6, 7, 8, 9, 10], Scarf potential [11, 12, 13], Woods-Saxon Potential [14, 15, 16, 17, 18, 19, 20], Hulthén Potential [21, 22, 23, 24], Manning-Rosen potential [25, 26], Rosen-Morse potential [10, 27, 28], etc. The bound state energy spectra of the Schrödinger equation of these potentials have been investigated by a variety of techniques [26, 29, 30, 31, 32], such as the factorization method, the super-symmetric quantum mechanics, Nikiforov-Uvarov method, Path integral solution, etc.

The motivation of the present paper is to solve the one-dimensional time-independent Schrödinger equation for Woods-Saxon plus Rosen-Morse potential. Nikiforov-Uvarov method [32] is used to solve the Schrödinger equation for this potential. We have obtained PT-symmetric [33, 34, 35, 36, 37, 38, 39, 40, 41, 42] and non PT-symmetric solutions. We have also obtained the eigenvalues and eigenfunctions of the Woods Saxon potential and the Rosen-Morse potential separately.

The organization of the present paper is as follows. After a brief introductory discussion of the Nikiforov-Uvarov method in Sec. II, we obtain the energy eigenvalues and eigenfunctions for real and complex cases of Woods-Saxon plus Rosen-Morse potential in Sec. III. In Sec. IV and Sec. V, we discuss the solution of PT-symmetric and non PT-symmetric Woods-Saxon plus Rosen-Morse potential. In Sec. VI and Sec. VII, we discuss, the eigenvalues and eigenfunctions of Woods-Saxon potential and Rosen-Morse potential respectively. Finally, conclusions and remarkable facts are discussed in the last section.

II. NIKIFOROV-UVAROV METHOD

The differential equations whose solutions are the special functions of hypergeometric type can be solved by using the Nikiforov-Uvarov method which has been developed by Nikiforov and Uvarov [32]. In this method, the one dimensional Schrödinger equation is reduced to an equation by an appropriate coordinate transformation, $s = s(x)$,

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a polynomial, at most of first degree. For find the particular solution to Eq. (1), we set the following wave function as a multiple of two independent parts

$$\psi(s)=\varphi(s)y(s). \tag{2}$$

With this substitution Eq. (1) reduces to an equation of hypergeometric type

$$\sigma(s)y''\psi(s)+\tau(s)y'(s)+\lambda y(s)=0, \tag{3}$$

provided the following conditions be satisfied:

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)}, \tag{4}$$

$$\tau(s)=\tilde{\tau}(s)+2\pi(s)\,,\ \tau'(s)<0. \tag{5}$$

The condition $\tau'(s)<0$ helps to generate energy eigenvalues and corresponding eigenfunctions. The condition $\tau'(s)>0$ has widely discussed in [32]. The λ in Eq. (3) satisfies the following second-order differential equation

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s), \tag{6}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. It is worthwhile to note that λ or are λ_n obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. The other part $y_n(s)$ of the wavefunction (2) is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation [43, 44, 45]

$$y_n(s)=\frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{7}$$

where C_n being the normalization constant and the weight function $\rho(s)$ satisfies the relation as

$$\frac{d}{ds}\left[\sigma(s)\rho(s)=\tau(s)\rho(s)\right]. \tag{8}$$

On the other hand, in order to find the eigenfunctions, $\varphi_n(s)$ and $y_n(s)$ in Eqs. (4) and (7) and eigenvalues λ_n in Eq. (6), we need to calculate the functions:

$$\pi(s)=\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)}\,, \tag{9}$$

$$k=\lambda-\pi'(s). \tag{10}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in (9) can be arranged to be the square of a polynomial of first degree [32]. This is possible only if its discriminant is zero. Thus, the value of k obtained from the equation (9) can substituted in equation (10). The energy eigenvalues are obtained from equations (6) and (10).

III. WOODS-SAXON PLUS ROSEN-MORSE POTENTIAL

The generalized Woods-Saxon plus Rosen-Morse potential is given by

$$\frac{d^2\psi}{dx^2}+\Big[E+V_1\frac{e^{-2ax}}{1+qe^{-2ax}}-V_2\frac{e^{-4ax}}{\left(1+e^{-2ax}\right)^2}+ \\ -V_3\sec h_q^{\,2}(ax)-V_3\tanh_q(ax). \tag{11}$$

Where the deformed hyperbolic functions is defined as:

$$\sinh_q x=\frac{e^x-qe^{-x}}{2}\ ,\ \cosh_q x=\frac{e^x+qe^{-x}}{2},$$
$$\tanh_q x=\frac{\sinh_q x}{\cosh_q x}.$$

The Schrödinger equation becomes

$$\frac{d^2\psi}{dx^2}+\Big[E+V_1\frac{e^{-2ax}}{1+qe^{-2ax}}-V_2\frac{e^{-4ax}}{\left(1+e^{-2ax}\right)^2}+ \\ V_3\sec h_q^{\,2}(ax)+V_4\tanh_q(ax)\Big]\psi=0, \tag{12}$$

where η=2m=1. Setting the following notations

$$\varepsilon=-\frac{E}{4a^2}\,,\beta_i=\frac{V_i}{4a^2}(>0)\,,i=1,2,3,4\,,s=e^{-2ax}, \tag{13}$$

with $\varepsilon>0(E<0)$ for bound states, Eq. (12) becomes

$$\frac{d^2\psi}{ds^2}+\frac{1-qs}{s-qs^2}\frac{d\psi}{ds}+\frac{1}{(s-qs^2)^2}\Big[-\Big\{q^2(\varepsilon-\beta_4)-$$

$$q(\beta_1-2q\beta_4)+\beta_2\Big\}s^2+\Big\{2q(\varepsilon-\beta_4)-$$

$$(\beta_1-2q\beta_4+4\beta_3)\Big\}s-(\varepsilon-\beta_4)\Big]\psi=0. \qquad (14)$$

After the comparison of Eq. (14) with Eq. (1), we have

$\tilde{\tau}(s)=1-qs$, $\sigma(s)=s-qs^2$ and

$$\tilde{\sigma}(s)=-\Big\{q^2(\varepsilon-\beta_4)-q(\beta_1-2q\beta_4)+\beta_2\Big\}s^2+$$

$$\Big\{2q(\varepsilon-\beta_4)-(\beta_1-2q\beta_4+4\beta_3)\Big\}s-(\varepsilon-\beta_4). \qquad (15)$$

Substituting these polynomials into Eq. (9), we have

$$\pi(s)=-\frac{qs}{2}\pm\frac{1}{2}\Big[(2\sqrt{\varepsilon-\beta_4}-\mu P)qs-2\sqrt{\varepsilon-\beta_4}\Big]$$

$$\text{If } k=-(\beta_1-2q\beta_4+4\beta_3)+\mu q\sqrt{\varepsilon-\beta_4}P, \qquad (16)$$

where μ = +1,-1 and

$$P=\sqrt{1+\frac{16\beta_3}{q}+\frac{4\beta_2}{q^2}}=\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}.$$

For bound state solutions, it is necessary to choose

$$\pi(s)=-\frac{qs}{2}-\frac{1}{2}\Big[(2\sqrt{\varepsilon-\beta_4}-\mu P)qs-2\sqrt{\varepsilon-\beta_4}\Big].$$

$$\text{If } \quad k=-(\beta_1-2q\beta_4+4\beta_3)+\mu q\sqrt{\varepsilon-\beta_4}P. \qquad (17)$$

The following track in this selection is to achieve the condition $\tau'(s)<0$. Therefore $\tau(s)$ becomes

$$\tau(s)=1+2\sqrt{\varepsilon-\beta_4}-[2+2\sqrt{\varepsilon-\beta_4}-\mu P]qs, \qquad (18)$$

and then its negative derivatives become

$$\tau'(s)=-[2+2\sqrt{\varepsilon-\beta_4}-\mu P]qs. \qquad (19)$$

Therefore from Eqs. (6) and (10) we have

$$\lambda=\lambda_n=n[2+2\sqrt{\varepsilon-\beta_4}-\mu P]q+n(n+1)q, \qquad (20)$$

and

$$\lambda=(\beta_1-2q\beta_4+4\beta_3)+\mu q\sqrt{\varepsilon-\beta_4}P$$

$$-\frac{q}{2}-\frac{q}{2}(2\sqrt{\varepsilon-\beta_4}-\mu P). \qquad (21)$$

Comparing Eqs. (20) and (21) we have

$$n[2+2\sqrt{\varepsilon-\beta_4}-\mu P]q+n(n+1)q=$$

$$-(\beta_1-2q\beta_4+4\beta_3)+\mu q\sqrt{\varepsilon-\beta_4}P$$

$$-\frac{q}{2}-\frac{q}{2}(2\sqrt{\varepsilon-\beta_4}-\mu P),$$

$$\Rightarrow(2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+n(n+1)+\frac{1}{2}-$$

$$\left(n+\frac{1}{2}\right)\mu P=-\frac{\beta_1-2q\beta_4+4\beta_3}{q},$$

$$\Rightarrow 4(2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+(2n+1-\mu P)^2=$$

$$-\frac{4(\beta_1-2q\beta_4+4\beta_3)}{q}-1+P^2, \qquad (20)$$

and

$$\lambda=(\beta_1-2q\beta_4+4\beta_3)+\mu q\sqrt{\varepsilon-\beta_4}P$$

$$-\frac{q}{2}-\frac{q}{2}(2\sqrt{\varepsilon-\beta_4}-\mu P). \qquad (21)$$

Comparing Eqs. (20) and (21) we have

$$n[2+2\sqrt{\varepsilon-\beta_4}-\mu P]q+n(n+1)q=$$

$$-(\beta_1-2q\beta_4+4\beta_3)+\mu q\sqrt{\varepsilon-\beta_4}P$$

$$-\frac{q}{2}-\frac{q}{2}(2\sqrt{\varepsilon-\beta_4}-\mu P)$$

$$\Rightarrow(2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+n(n+1)+\frac{1}{2}-$$

$$\left(n+\frac{1}{2}\right)\mu P=-\frac{\beta_1-2q\beta_4+4\beta_3}{q}$$

$$\Rightarrow 4(2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+(2n+1-\mu P)^2=$$

$$-\frac{4(\beta_1-2q\beta_4+4\beta_3)}{q}-1+P^2. \qquad (22)$$

Substituting the values of β_1, β_2, β_3, β_4 ε and P we obtain the energy eigenvalues:

$$E_n=-\frac{a^2}{4}\left[\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{4V_2}{q^2a^2}}\right)-\right.$$

$$\left.\frac{\frac{V_2}{q^2a^2}-\frac{V_1-2qV_4}{qa^2}}{\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{4V_2}{q^2a^2}}\right)}\right]^2-V_4, \quad (23)$$

$$n=0,1,2,................, q\geq 1..$$

Again Eq. (22) can be expressed as

$$(2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+n(n+1)+\frac{1}{2}-$$

$$\left(n+\frac{1}{2}\right)\mu P=-\frac{\beta_1-2q\beta_4+4\beta_3}{q}$$

$$\Rightarrow (2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+n(n+1)+\frac{1}{2}-$$

$$\frac{1}{2}\left(2n+1-\mu P\right)\left(2n+1\right)-\frac{1}{2}\left(2n+1\right)^2$$

$$=-\frac{\beta_1-2q\beta_4+4\beta_3}{q},$$

$$\Rightarrow (2n+1-\mu P)\sqrt{\varepsilon-\beta_4}+\left(2n+1-\mu P\right)(2n+1)$$

$$=n(n+1)-\frac{\beta_1-2q\beta_4+4\beta_3}{q}, \quad (24)$$

$$E_n=-4a^2\left[\frac{2n+1}{2}-\frac{n(n+1)-\frac{V_1-2qV_4+4V_3}{4qa^2}}{2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}}\right]^2-V_4,$$

where $n=0,1,2,................, q\geq 1.$ (25)

From the Eqs. (5), (8) and (15) we obtain the weight function

$$\rho(s)=s^{2\sqrt{\varepsilon-\beta_4}}(1-qs)^{-\mu P}, \quad (13)$$

and from Eqs. (4), (15), and (17) we have

$$\varphi(s)=s^{\sqrt{\varepsilon-\beta_4}}(1-qs)^{\frac{1}{2}(1-\mu P)}. \quad (27)$$

Now using the properties of Jacobi Polynomial [43, 45],

$$P_n^{(c,d)}(x)=\frac{(-1)^n(1-x)^{-c}(1+x)^{-d}}{2^n n!}\times$$

$$\frac{d^n}{dx^n}\left[(1-x)^{n+c}(1+x)^{n+d}\right],$$

we have

$$P_n^{(2\sqrt{\varepsilon-\beta_4},-\mu P)}(1-2qs)=\frac{(-2q)^n s^{2\sqrt{\varepsilon-\beta_4}}(1-qs)^{-\mu P}}{n!}\times$$

$$\frac{d^n}{dx^n}\left[s^{n+2\sqrt{\varepsilon-\beta_4}}(1-qs)^{n-\mu P}\right]. \quad (28)$$

The wave functions are obtained from Eqs. (2), (7), (25-27)

$$\psi_n(s)=N_n s^{\sqrt{\varepsilon-\beta_4}}(1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{4V_2}{q^2a^2}}\right)}\times$$

$$P_n^{\left(2\sqrt{\varepsilon-\beta_4},-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{4V_2}{q^2a^2}}\right)}(1-2qs), \quad (29)$$

where the normalization constant N_n satisfying the normalization condition

$$\int_0^1\left|\psi_n(s)\right|^2 ds=1.$$

Two different forms of Jacobi Polynomials [43, 45] are

$$P_n^{(c,d)}(x)=2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+c}{p}\binom{n+d}{n-p}(1-x)^{-c}(1+x)^{-d}, \quad (31)$$

$$P_n^{(c,d)}(x)=\frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\left(\frac{x-1}{2}\right)^r, \quad (32)$$

where $\binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$.

From Eq.(31) and Eq.(32), we have

$$P_n^{(2c,2d)}(1-2qs)=(-1)^n\Gamma(n+2c+1)\Gamma(n+2d+1)\times$$

$$\sum_{p=0}^{n}\frac{(-1)^p q^{n-p}s^{n-p}(1-qs)^p}{p!(n-p)!\Gamma(n+2d+1)\Gamma(n+2c-p+1)}, \quad (33)$$

$$P_n^{(2c,2d)}(1-2qs) = (-1)^n \frac{\Gamma(n+2c+1)}{\Gamma(n+2c+2d+1)} \times \sum_{r=0}^{n} \frac{(-1)^r q^r \Gamma(n+2c+2d+1)}{r!(n-r)!\Gamma(2c+r+1)} s^r. \quad (34)$$

Using the Eqs. (33) and (34), we have

$$1 = N_q^2 (-1)^n \frac{[\Gamma(n+2c+1)]^2 \Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times$$

$$\sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r} \Gamma(n+2c+2d+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)} \Big\{ \frac{I_{nq}(p,r)}{\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \Big\}, \quad (35)$$

where

$$I_{nq}(p,r) = \int_0^1 s^{n+2\sqrt{\varepsilon-\beta_4}+r-p} (1-qs)^{p-\mu P+1} ds$$

$$c = \sqrt{\varepsilon - \beta_4}, d = -\frac{1}{2}\mu P, P = \sqrt{1 + \frac{4V_3}{qa^2} + \frac{4V_2}{q^2 a^2}}. \quad (36)$$

Using the following integral of hypergeometric function

$$\int_0^1 s^{a-1}(1-s)^{c-a-1}(1-qs)^{-b} ds = {}_2F_1(a,b;c;q) \times \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}. \quad (37)$$

Provided $\mathrm{Re}(c) > \mathrm{Re}(a) > 0, \left|\arg(1-q)\right| < \pi$

$$\int_0^1 s^{a-1}(1-qs)^{-b} ds = \frac{{}_2F_1(a,b;a+1;q)}{a}. \quad (38)$$

Using Eq. (35) and Eq.(37) we have

$$I_{nq}(p,r) = \frac{1}{\left(n + 2\sqrt{\varepsilon-\beta_4} + r - p + 1\right)} \times {}_2F_1(n + 2\sqrt{\varepsilon-\beta_4} + r - p + 1, \mu P - p - 1; n + 2\sqrt{\varepsilon-\beta_4} + r - p + 2, q). \quad (39)$$

IV. *PT*-SYMMETRIC NON-HERMITIAN CASE

In general, when a Hamiltonian commutes with PT, it is called PT-symmetric Hamiltonian, where the parity operator P and time reversal operator T, satisfying the following relations

$$PxP^{-1} = -x,\ PpP^{-1} = -p = TpT^{-1},\ TAT^{-1} = -iA,$$

In this case, we set the potential parameters in Eq. (11) as V_1, V_2, V_3, V_4, $q \in \measuredangle$ and $a \in \bigcap \measuredangle$ $(a \to ia)$ (Where $\measuredangle$ and $\bigcap \measuredangle$ are the set of purely real numbers and purely complex numbers respectively), then Eq. (11) becomes

$$V(x) = V_R(x) + iV_I(x),$$

$$= \frac{1}{\left(1+q^2+2q\cos 2ax\right)^2} \times \Big[-V_1(\cos 2ax + q)(1+q^2+2q\cos 2ax) + V_2(q^2 + 2q\cos 2ax + \cos 4ax) - 4V_3(2q + (q^2-1)\cos 2ax) - V_4(1-q^2)(1+q^2+2q\cos 2ax)\Big]$$

$$+ \frac{i\sin 2ax}{\left(1+q^2+2q\cos 2ax\right)^2} \times \Big[V_1(1+q^2+2q\cos 2ax) - 2V_2(q+\cos 2ax) - 4V_3(q^2-1) - 2qV_4(1+q^2+2q\cos 2ax)\Big]. \quad (40)$$

Then $V(x)$ satisfies the relation $(PT)V(x)(PT)^{-1} = V(x)$. The energy eigenvalues of the potential (40) are

$$E_n = 4a^2 \left[\frac{2n+1}{2} - \frac{n(n+1) + \dfrac{V_1 - 2qV_4 + 4V_3}{4qa^2}}{2n+1-\mu\sqrt{1 - \dfrac{4V_3}{qa^2} - \dfrac{V_2}{q^2 a^2}}} \right]^2 - V_4 \quad (41)$$

where $q \geq 1$ and the condition for n is

$$n < \frac{1}{2}\sqrt{\frac{V_1 - 2qV_4}{qa^2} - \frac{V_2}{q^2 a^2}} + \frac{\mu}{2}\sqrt{1 - \frac{4V_3}{qa^2} - \frac{V_2}{q^2 a^2}} - \frac{1}{2}. \quad (42)$$

The corresponding eigenfunctions are

$$\psi_n(s) = A_n s^{\sqrt{\varepsilon-\beta_4}} (1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1-\frac{4V_3}{qa^2}-\frac{V_2}{q^2a^2}}\right)} \times$$

$$P_n^{(2\sqrt{\varepsilon-\beta_4}, -\mu\sqrt{1-\frac{4V_3}{qa^2}-\frac{V_2}{q^2a^2}})}(1-2qs), \qquad (43)$$

where A_n is normalization constant satisfying the relation

$$1 = A_n^2(-1)^n \frac{[\Gamma(n+2c+1)]^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times$$

$$\sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r}\Gamma(n+2c+2d+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)} \left\{ \frac{I_{nq}(p,r)}{\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \right\}. \qquad (44)$$

With

$$I_{nq}(p,r) = \frac{1}{\left(n+2\sqrt{\varepsilon-\beta_4}+r-p+1\right)} \times$$

$${}_2F_1(n+2\sqrt{\varepsilon-\beta_4}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon-\beta_4}+r-p+2, q),$$

where

$$c = \sqrt{\varepsilon-\beta_4}, d = -\frac{1}{2}\mu P, P = \sqrt{1-\frac{4V_3}{qa^2}-\frac{4V_2}{q^2a^2}}. \qquad (45)$$

V. NON *PT*-SYMMETRIC NON-HERMITIAN CASE

Now let us take the potential parameters as consider the case, V_2, $V_4 \in \measuredangle$, and V_1, V_3, $a \in \cap \measuredangle$ $(V_1 \to iV_1, V_3 \to iV_3, q \to iq, a \to ia)$. Then potential (11) takes the form

$$V(x) = V_R(x) + iV_I(x)$$

$$= \frac{1}{\left(1+q^2+2q\sin 2ax\right)^2} \times \Big[-V_1(\sin 2ax+q)(1+q^2+2q\sin 2ax)$$
$$-V_2(q^2+2q\sin 2ax+\cos 4ax)$$
$$-4V_3(q^2-1)\cos 2ax$$
$$-2qV_4\cos 2ax(1+q^2+2q\sin 2ax)\Big]$$
$$+\frac{i}{\left(1+q^2+2q\sin 2ax\right)^2} \times \Big[-V_1\cos 2ax(1+q^2+2q\cos 2ax)$$
$$-V_2(2q\cos 2ax+\sin 4ax)$$
$$-4V_3(q^2+1)(\sin 2ax+2q)$$
$$+2qV_4\cos 2ax(1+q^2+2q\sin 2ax)\Big]. \qquad (46)$$

This kind of potential is non PT -symmetric. The energy eigenvalues are

$$E_n = 4a^2\left[\frac{2n+1}{2} - \frac{n(n+1)+\frac{V_1-2qV_4+4V_3}{4qa^2}}{2n+1-\mu\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}}\right]^2 - V_4, \qquad (47)$$

condition for n is

$$n < \frac{1}{2}\sqrt{\frac{V_1-2qV_4}{qa^2}+\frac{V_2}{q^2a^2}} + \frac{\mu}{2}\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}} - \frac{1}{2}. \qquad (48)$$

The corresponding eigenfunctions are

$$\psi_n(s) = B_n s^{\sqrt{\varepsilon-\beta_4}}(1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\right)} \times$$

$$P_n^{(2\sqrt{\varepsilon-\beta_4}, -\mu\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}})}(1-2qs), \qquad (49)$$

Where B_n is normalization constant satisfying the relation

$$1 = B_n^2(-1)^n \frac{[\Gamma(n+2c+1)]^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times$$

$$\sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r}\Gamma(n+2c+2d+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)} \left\{ \frac{I_{nq}(p,r)}{\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \right\}, \qquad (50)$$

with

$$I_{nq}(p,r)=\frac{1}{\left(n+2\sqrt{\varepsilon-\beta_4}+r-p+1\right)}\times$$

$${}_2F_1(n+2\sqrt{\varepsilon-\beta_4}+r-p+1,\mu P-p-1;$$
$$n+2\sqrt{\varepsilon-\beta_4}+r-p+2,q),$$

where

$$c=\sqrt{\varepsilon-\beta_4},d=-\frac{1}{2}\mu P,P=\sqrt{1-\frac{4V_3}{qa^2}+\frac{4V_2}{q^2a^2}}. \qquad (51)$$

VI. WOODS-SAXON POTENTIAL

Setting $V_3 = V_4=0$, the potential (11) becomes the Woods-Saxon potential[14, 15, 16, 17, 18, 19, 20]

$$V(x)=-V_1\frac{e^{-2ax}}{1+qe^{-2ax}}+V_2\frac{e^{-4ax}}{(1+qe^{-2ax})^2}. \qquad (52)$$

The energy eigenvalues and wave functions of this potential are obtained from Eqs. (23), (29) setting

$$\mu=1 \quad E_n=-\frac{a^2}{4}\left[\left(2n+1-\sqrt{1+\frac{4V_2}{q^2a^2}}\right)-\frac{V_2-qV_1}{q^2a^2\left(2n+1-\sqrt{1+\frac{4V_2}{q^2a^2}}\right)}\right]^2, \qquad (53)$$

$$n=0,1,2,................,q\geq 1,$$

$$\psi_n(s)=C_n s^{\sqrt{\varepsilon}}(1-qs)^{\frac{1}{2}\left(1-\sqrt{1+\frac{4V_2}{q^2a^2}}\right)}\times P_n^{\left(2\sqrt{\varepsilon},-\sqrt{1+\frac{4V_2}{q^2a^2}}\right)}(1-2qs), \qquad (54)$$

where the normalization constant D_n satisfying the normalization condition

$$1=D_n^2(-1)^n\frac{[\Gamma(n+2c+1)]^2\,\Gamma(n+2d+1)}{\Gamma(2c+2d+1)}\times$$

$$\sum_{p,r=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)}\left\{\frac{I_{nq}(p,r)}{\Gamma(r+2d+1)\Gamma(n+2c-p+1)}\right\}, \qquad (55)$$

with

$$I_{nq}(p,r)=\frac{1}{\left(n+2\sqrt{\varepsilon}+r-p+1\right)}\times$$

$${}_2F_1(n+2\sqrt{\varepsilon}+r-p+1,\mu P-p-1;$$
$$n+2\sqrt{\varepsilon}+r-p+2,q),$$

where $c=\sqrt{\varepsilon},d=-\frac{1}{2}P,P=\sqrt{1+\frac{4V_2}{q^2a^2}}$. (56)

We are now going to consider different forms of generalized Woods-Saxon potential,viz at least one of the parameters is purely imaginary. When $a\to ia$ and $V_1,V_2,q\in\measuredangle$, then $V(x)$ becomes

$$V(x)=V_R(x)+iV_I(x)$$
$$=\left[-V_1\frac{\cos 2ax+q}{\left(1+q^2+2q\cos 2ax\right)}+V_2\frac{q^2+2q\cos 2ax+\cos 4ax}{\left(1+q^2+2q\cos 2ax\right)^2}\right]$$
$$+i\left[V_1\frac{\sin 2ax}{\left(1+q^2+2q\cos 2ax\right)}-V_2\frac{2q\sin 2ax+\sin 4ax}{\left(1+q^2+2q\cos 2ax\right)^2}\right], \qquad (57)$$

Then $(PT)V(x)(PT)^{-1}=V(x)$. The real positive energy eigenvalues are given by

$$E_n=\frac{a^2}{4}\left[\left(2n+1-\sqrt{1-\frac{4V_2}{q^2a^2}}\right)-\right.$$

$$\left.\frac{qV-V_2}{q^2a^2\left(2n+1-\sqrt{1-\frac{4V_2}{q^2a^2}}\right)}\right]^2, \quad (58)$$

if and only if

$$n=0,1,...<\frac{1}{2}\sqrt{\frac{V_1}{qa^2}-\frac{V_2}{q^2a^2}}+\frac{1}{2}\sqrt{1-\frac{V_2}{q^2a^2}}-\frac{1}{2}. \quad (59)$$

The Eq. (58) is consistent with Eq. (59) of [17] for $\hbar$=2m=1 and $a=\frac{\alpha_1}{2}$. The eigenvalues are always positive real when V_2 = 0and condition for n is $n=0,1,...<\frac{1}{2}\sqrt{\frac{V_1}{qa^2}}-1$ but can be complex for $V_2 > q^2\, a^2$. Next we set $V_1 \to iV_1$, $V_2 \in \measuredangle$, $a \to ia$ and $q \in \measuredangle$, then Eq. (11) takes the form

$$V(x)=V_R(x)+iV_I(x)$$

$$=\left[-V_1\frac{\sin 2ax+q}{\left(1+q^2+2q\sin 2ax\right)}\right.$$

$$\left.-V_2\frac{q^2+2q\sin 2ax+\cos 4ax}{\left(1+q^2+2q\sin 2ax\right)^2}\right]$$

$$-i\left[V_1\frac{\cos 2ax}{\left(1+q^2+2q\sin 2ax\right)}+\right.$$

$$\left.+V_2\frac{2q\cos 2ax+\sin 4ax}{\left(1+q^2+2q\sin 2ax\right)^2}\right]. \quad (60)$$

Such a potential is called non-PT -symmetric potential. The complex energy eigenvalues are given by

$$E_n=4a^2\left[\frac{2n+1}{2}-\frac{n(n+1)+i\frac{V_1}{4qa^2}}{2n+1+\sqrt{1-\frac{V_2}{q^2a^2}}}\right]^2,$$

$$n\geq 0, q\geq 1. \quad (61)$$

VII. ROSEN-MORSE POTENTIAL

Assuming $V_1=V_2=0$, the potential (11) turns into Rosen Morse potential [11]

$$V(x)=-V_3\sec h_q{}^2(ax)-V_4\tanh_q(ax). \quad (62)$$

The energy eigenvalues and wave functions of this potential by setting $\mu=-1$

$$E_n=-\frac{a^2}{4}\left(2n+1+\sqrt{1+\frac{4V_3}{qa^2}}\right)^2$$

$$-\frac{V_4{}^2}{a^2}\left(2n+1+\sqrt{1+\frac{4V_3}{qa^2}}\right)^{-2}, \; n\geq 0, q\geq 1. \quad (63)$$

Which is consistent with [28]. The eigenfunctions are

$$\psi_n(s)=L_n s^{\sqrt{\varepsilon-\beta_4}}(1-qs)^{\frac{1}{2}\left(1+\sqrt{1+\frac{4V_3}{qa^2}}\right)}\times$$

$$P_n^{\left(2\sqrt{\varepsilon-\beta_4},+\sqrt{1+\frac{4V_3}{qa^2}}\right)}(1-2qs), \quad (64)$$

where L_n ,the normalization constant is given by

$$1=L_n^2(-1)^n\frac{[\Gamma(n+2c+1)]^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)}\times$$

$$\sum_{p,r=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)}\left\{\frac{I_{nq}(p,r)}{\Gamma(r+2d+1)\Gamma(n+2c-p+1)}\right\}, \quad (65)$$

with

$$I_{nq}(p,r)=\frac{1}{\left(n+2\sqrt{\varepsilon}+r-p+1\right)}\times$$

$${}_2F_1(n+2\sqrt{\varepsilon}+r-p+1,\mu P-p-1;$$

$$n+2\sqrt{\varepsilon}+r-p+2,q),$$

$$c=\sqrt{\varepsilon}, d=\frac{1}{2}P, P=-\sqrt{1+\frac{4V_3}{qa^2}}. \quad (66)$$

We are now going to consider different forms of generalized Rosen-Morse potential, viz at least one of the parameters is purely imaginary. When $a \to ia$ and $V_3, V_4, q \in \measuredangle$, then $V(x)$ becomes

$$V(x) = V_R(x) + iV_I(x),$$

$$= \left[-4V_3 \frac{2q + (q^2+1)\cos 2ax}{\left(1+q^2+2q\cos 2ax\right)^2} - V_4 \frac{1-q^2}{\left(1+q^2+2q\cos 2ax\right)} \right]$$

$$+ i\left[-4V_3 \frac{(q^2-1)\sin 2ax}{\left(1+q^2+2q\cos 2ax\right)^2} - V_4 \frac{2q\sin 2ax}{\left(1+q^2+2q\cos 2ax\right)} \right]. \quad (67)$$

Then $V(x)$ satisfies the relation $(PT)V(x)(PT)^{-1} = V(x)$. The positive energy eigenvalues are then given by

$$E_n = \frac{a^2}{4}\left(2n+1-\sqrt{1-\frac{4V_3}{qa^2}}\right)^2 + \frac{V_4^{\,2}}{a^2}\left(2n+1-\sqrt{1-\frac{4V_3}{qa^2}}\right)^{-2}, \; n \ge 0, q \ge 1. \quad (68)$$

Again for $V_3, a, q \in \cap \measuredangle$, $V_4 \in \measuredangle$, the potential (62) is non-PT-symmetric potential. The potential is given by

$$V(x) = V_R(x) + iV_I(x)$$

$$= \left[4V_3 \frac{(1-q^2)\cos 2ax}{\left(1+q^2+2q\sin 2ax\right)^2} - V_4 \frac{1-q^2}{\left(1+q^2+2q\sin 2ax\right)} \right]$$

$$+ i\left[-4V_3 \frac{(1+q^2)(\sin 2ax + 2q)}{\left(1+q^2+2q\sin 2ax\right)^2} + V_4 \frac{2q\cos 2ax}{\left(1+q^2+2q\sin 2ax\right)} \right]. \quad (69)$$

The positive energy eigenvalues are then given by

$$E_n = \frac{a^2}{4}\left(2n+1-\sqrt{1+\frac{4V_3}{qa^2}}\right)^2 - \frac{V_4^{\,2}}{a^2}\left(2n+1-\sqrt{1+\frac{4V_3}{qa^2}}\right)^{-2}, \; n \ge 0, q \ge 1. \quad (70)$$

VIII. CONCLUSION

In this paper, the Schrödinger equation with Woods-Saxon plus Rosen-Morse potential have been solved by using the Nikiforov-Uvarov method. Some interesting results including complex *PT*-symmetric and non-*PT*-symmetric versions of the Woods-Saxon potential and the Rosen-Morse potential have also been discussed. Energy eigenvalues for the Woods-Saxon potential and the Rosen-Morse potential have been presented separately. It is shown that the results are in good agreement with the ones obtained by others.

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