The equivalent expressions between escape velocity and orbital velocity

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Abstract

In this paper, we derived some equivalent expressions between both, escape velocity and orbital velocity of a body that orbits around of a central force, from which is possible to derive the Newtonian expression for the Kepler's third law for both, elliptical or circular orbit. In addition, we found that the square of the escape velocity multiplied by the escape radius is equivalent to the square of the orbital velocity multiplied by the orbital diameter. Extending that equivalent expression to the escape velocity of a given black hole, we derive the corresponding analogous equivalent expression. We substituted in such an expression the respective data of each planet in the Solar System case considering the Schwarzschild radius for a central celestial body with a mass like that of the Sun, calculating the speed of light in all the cases, then confirming its applicability.

Keywords: Escape velocity, Orbital velocity, Newtonian gravitation, Black hole, Schwarzschild radius.

Resumen

En este trabajo, derivamos algunas expresiones equivalentes entre la velocidad de escape y la velocidad orbital de un cuerpo que orbita alrededor de una fuerza central, de las cuales se puede derivar la expresión Newtoniana para la tercera ley de Kepler para un cuerpo en órbita elíptica o circular. Además, encontramos que el cuadrado de la velocidad de escape multiplicada por el radio de escape es equivalente al cuadrado de la velocidad orbital multiplicada por el diámetro orbital. Extendiendo esa expresión equivalente a la velocidad de escape de un agujero negro dado, derivamos la expresión equivalente análoga correspondiente. Substituimos en tal expresión equivalente los respectivos datos de cada planeta en el caso del Sistema Solar considerando el radio de Schwarzschild para un cuerpo celeste central con una masa como la del Sol, calculando la velocidad de la luz en todos los casos, confirmando así su aplicabilidad.

Palabras clave: Velocidad de escape, Velocidad orbital, Gravitación Newtoniana, Agujero negro, Radio de Schwarzschild.

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I. INTRODUCTION

Several mathematical relations in physics and their variables can be related between them by equivalent expressions, which link comparatively different aspects of physical phenomena, such as the escape velocity and the orbital velocity of a body orbiting around of a massive celestial body, which constitutes the central force of a gravitational system as a planet, the Sun, a star and even a black hole, where both velocities are related with the gravitational potential energy and with the conservation of energy in a gravitational system. Such equivalent expressions have served as mathematical tools that allow to improve a detailed comparative analysis between parameters which describe the physical phenomena, such as the fields, the forces and the interactions, among others; reason for which we can raise the importance of the equivalent expressions and their contribution in the analysis and the theoretical development that explain the dynamic interaction between the bodies, even though some of these expressions have been derived from algebraic equivalences and sometimes they have not an intrinsic physical referent or meaning nothing more than be a tool which relates the physical phenomena and their variables in a numerical or comparative way.

In this paper we review both, classical escape velocity and orbital velocity of a body that orbits around of a central force. Such a review allows one to derive some equivalent expressions between both velocities, from which is possible to derive the Newtonian expression for the Kepler's third law for a body in both, elliptical or circular orbit. We derive some equivalent expressions between both velocities, finding that square of escape velocity multiplied by the escape radius is equivalent to the square of orbital velocity multiplied by the orbital diameter. Extending that equivalent expression to the escape velocity in a given black hole, which is related with the Schwarzschild radius, we derived corresponding analogous equivalent expression between both, the escape velocity of a black hole (it is, the speed of light) and the orbital velocity of a circular orbit around of such a black hole. In order to confirm the applicability of such an equivalent expression, by substituting in that expression the respective data of each planet in the Solar System case considering the Schwarzschild radius for a celestial body with a mass like that of the Sun, we calculate the speed of light in all the cases. Thus, considering that the algebraic expressions are equivalents if their final values obtained by substituting the respective values of the variables always are the same, we confirm their applicability. Even though some of the here derived equivalent expressions do not have an intrinsic physical referent or meaning, the terms related in the here derived equivalent expressions can be also applied by substitution to calculate new equivalent expressions of the known mathematical expressions, including those expressions which relates the speed of the light in a gravitational system of two bodies.

II. THE CLASSICAL ESCAPE VELOCITY REVISITED

The **escape velocity** is considered the minimum speed in a radial direction that a body or projectile would have to be moving when it reaches a point in space at a radial distance from the center of mass of a celestial body in order to escape of the gravitational force of such a celestial body [1, 2]. That means that the body or projectile which is sent up with such a speed will not return to fall on such a celestial body, being in rest with a null velocity to a sufficiently great distance (infinite in principle) of such a celestial body [1]. Since relative velocity is the velocity of one body with respect to another, relative escape velocities is related only in systems of two bodies.

As known, the phenomenon of escape velocity is a consequence of conservation of energy of a body being projected upward against the downward gravitational force, from the center of mass of a cosmological body point of view. Thus, the escape velocity does not depend on the mass of the body or projectile, but it depends on the form of the gravitational potential energy of the body at any radial distance from the center of mass of a celestial body. For a body in motion with a given total energy, which is moving subject to conservative force it is only possible for the body to reach combinations of places and speeds which have that total energy, and places which have a higher potential energy than this cannot be reached at all.

The Law of conservation of energy states that the total of the body's potential and kinetic energy is a constant. In the rotating coordinate system, the expression of total energy [1] on the orbital plane is defined as

$$E = E_K + V(r), \tag{1}$$

where E_K is the kinetic energy and $V(r)$ is the potential energy.

For a gravitational system, E_K corresponds to the Newtonian kinetic energy of a body of mass m and $V(r)$ corresponds to the gravitational potential energy of such a mass at any radial distance r from the center of mass of the celestial body. By comparing the potential and kinetic energy values at some given point with the values at infinity, it is possible to determine the escape velocity equation. Thus, when total energy is zero, it gives the minimum energy required for the body to escape to an infinite distance from the gravity of the celestial body of mass M. From expression (1), by conservation of energy, we get

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = 0, \tag{2}$$

where v is the velocity of the body of mass m and G is the gravitational constant. Solving equation (2) for square of velocity, then gives

$$v_e^2 = \frac{2GM}{r_e}, \tag{3}$$

where v_e is the escape velocity and r_e is the escape radius from the center of mass of the celestial body.

III. THE ORBITAL VELOCITY AND SOME OF ITS EQUIVALENCES WITH THE ESCAPE VELOCITY

If a body attains escape velocity, but is not directed straight away from the celestial body (or such a body arrives from outside and is captured by the gravitational force), then it will follow a curved path. Although that curved path does not form a closed shape, it is still considered an orbit whose focus is located at the center of mass of the celestial body. Assuming that gravity is the only significant force in the system, this body's velocity at any point in the orbit will be proportional to the escape velocity at that point due to the conservation of energy. Thus, when total energy is minor than zero, by his respective balance with the kinetic energy and the gravitational potential energy in the gravitational system, as well as by its constant of the angular momentum, then the body remains orbiting around of the celestial body [1].

According to the first Kepler's law [3], the orbit of every planet is an ellipse with the Sun at one focus.

We can generalize the first Kepler's law for any celestial body which constitutes the central force of a gravitational system with its center of mass at the focus. Thus, considering that a body is orbiting around of a celestial body, the distance of maximum approach from the center of mass is r_1 and the one of maximum distance is r_2, where $r_1 < r_2$, so that the velocities which the body has in these two extreme positions are v_1 and v_2, respectively, where $v_1 > v_2$. The constant of angular moment and of the energy allows one to relate these four magnitudes, giving

$$mr_1v_1 = mr_2v_2, \tag{4}$$

and

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2}. \tag{5}$$

Dividing both expressions (4) and (5) by the mass m, then finding out the velocity v_1 from equation (4), substituting in (5) and rearranging, yields

$$v_2^2 = \frac{2GMr_1}{r_2(r_1 + r_2)}. \tag{6}$$

In the same way, dividing both expressions (4) and (5) by the mass m, then finding out the velocity v_2 from equation (4), substituting in (5) and rearranging, yields

$$v_1^2 = \frac{2GMr_2}{r_1(r_1 + r_2)}. \tag{7}$$

According to the equation (5), each one of the two extreme positions (r_1 and r_2) has a particular potential energy, which changes in relation to the distance during the motion of the body along of the elliptical trajectory. A graphic of the potential energy is commonly performed [4] to show the kind of motion that a body can take along to the elliptical trajectory according to their particular energy value.

From expressions (4) and (5), we can derive the velocity v_2 in function of radius r_1 and its respective velocity v_1, giving

$$v_{\mathbf{2}} = \frac{2GM}{r_1 v_{\mathbf{1}}} - v_{\mathbf{1}}. \tag{8}$$

Thus, applying the equivalence from equation (3), we can express the velocity v_2 in terms of the escape velocity, giving

$$v_{\mathbf{2}} = \frac{v_e^2}{v_{\mathbf{1}}} - v_{\mathbf{1}}. \tag{9}$$

Substituting equivalence (9) in expression (4), we get

$$r_1 v_1 = r_2\left(\frac{v_e^2}{v_{\mathbf{1}}} - v_{\mathbf{1}}\right), \tag{10}$$

and finding out for square of escape velocity multiplied by the radius r_2, yields

$$v_e^2 r_2 = v_1^2 (r_1 + r_2). \tag{11}$$

Deriving the ratio between v_1 and v_2 from equivalence (9) and replacing in expression (4), we can obtain the radial distance r_2 from the center of mass, giving

$$r_{\mathbf{2}} = \frac{r_{\mathbf{1}} v_{\mathbf{1}}}{v_{\mathbf{2}}} = \frac{r_{\mathbf{2}} v_1^2}{v_e^2 - v_1^2}. \tag{12}$$

Furthermore, the addition of the radial distances r_1 and r_2 is equal to the major axis of the elliptical orbit, so that dividing the major axis by two, semi-major axis is obtained as

$$a = \frac{r_1 + r_2}{2}. \tag{13}$$

Thus, substituting expression (13) in equivalence (11), and then equaling to the equation (2), we get

$$v_e^2 r_e = v_1^2 2a = 2GM, \tag{14}$$

where finding out for the orbital velocity, yields

$$v = \sqrt{\frac{GM}{a}}. \tag{15}$$

which is the **orbital velocity** of a body in elliptical orbit around of a celestial body.

The transverse orbital velocity is inversely proportional to the distance to the central body because of the law of conservation of angular moment, or equivalently, Kepler's second law which affirms that an imaginary line joining a planet and the Sun sweeps out equal areas during equal intervals of time, this states that as a body moves around its orbit during a fixed amount of time, the line from the center of mass to the body sweeps a constant area of the orbital plane, regardless of which part of its orbit the body traces an ellipse during that period of time. This means that the body moves faster near its perihelion than near its aphelion, because at the smaller distance it needs to trace a greater arc to cover the same area. For elliptical orbits with small eccentricity, the length of the orbit is close to that of a circular one, and the mean orbital velocity can be approximated either from observations of the orbital period T and the semi-major axis of its orbit. Approximated orbital velocity for elliptical orbits with small eccentricity is given by

$$v \approx \frac{2\pi a}{T}, \tag{16}$$

Thus, equaling expressions (15) and (16), we get the Newtonian equivalence of GM term with respect to the period T for an elliptical orbit, defined as

$$GM = \frac{(2\pi)^2 a^3}{T^2}, \tag{17}$$

which contains the third Kepler's law, that affirms that the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit. Nevertheless, due to the nonzero planetary masses and perturbations in the planet orbits, Kepler's laws apply only approximately and not exactly to the motions in the Solar System.

Furthermore, although most orbits are elliptical in nature, a special case is the circular orbit. In this case, we can consider the orbital motion of a body in a circular orbit around a central force, rather than elliptical orbit; it is when the radius of the circumference is $r_o = r_1 = r_2$, and its respective velocity is $v_o = v_1 = v_2$. Thus, substituting in equation (8), yields

$$v_o = \frac{2GM}{r_o v_o} - v_o, \quad (18)$$

and finding out from (18) for square of the velocity, then square of orbital velocity for a circular orbit is given by

$$v_o^2 = \frac{GM}{r_o}, \quad (19)$$

where v_o is also called circular velocity of a body at a given point of a circular orbit around a central force in Newtonian gravitation [1], and the radius r_o is the radial distance from the central force. In the circular motion, circular velocity is given by

$$v_o = \frac{2\pi r_o}{T}, \quad (20)$$

and having the Newtonian equivalence of GM term with respect to the period T for a circular orbit, defined as

$$GM = \frac{(2\pi)^2 r_o^3}{T^2}, \quad (21)$$

we can substitute square of velocity from expression (20) in equation (21), then square of orbital velocity of a body in a circular orbit around of a central force is once again given by expression (19).

For a circular orbit, where the radius of the circumference is $r_o = r_1 = r_2$, equivalence (11) takes the form

$$v_e^2 r_e = v_o^2 (r_o + r_o) = 2v_o^2 r_o = v_o^2 D_o, \quad (22)$$

where D_o is the diameter of the circular orbit. Thus, expression (22) indicates that square of escape velocity multiplied by the escape radius is equivalent to the square of orbital velocity multiplied by the orbital diameter. In order to confirm the applicability of this equivalent expression (22) by evaluating its variables, we find out expression for escape radius, giving

$$r_e = \frac{2v_o^2 r_o}{v_e^2}. \quad (23)$$

Thus, having the data of the Sun-Earth system [5, 6], for instance, where Earth's mean orbital velocity equals to 29.8 km/sec, mean Sun-Earth distance equals to 149.60 10^6 km, and Sun escape velocity equals to 617.7 km/sec, then substituting such data in equivalent expression (23), calculated escape radius equals to 696,368.17 km, which is approximately the mean Sun radius, confirming the applicability of this equivalent expression.

IV. THE ESCAPE VELOCITY OF A BLACK HOLE AND ITS EQUIVALENCE WITH THE ORVITAL VELOCITY

As background, a **black hole** is a phase in the evolution of a star that has collapsed on itself, such that its gravitational force is so strong that not even light can escape its pull [7, 8]. Thus, it is called a "black hole" because light cannot escape from it, and it is as appears to telescopes. The idea of a body so massive that even light could not escape was first proposed by John Michell in 1783. In 1796, Laplace promoted the same idea of the so-called "dark star" [9]. In 1916, Karl Schwarzschild derived what is called the Schwarzschild radius from Einstein's gravitational field equations in the General Theory of Relativity [10]. It represents the event horizon of a black hole or the limiting radius. The event horizon or Schwarzschild radius is the defining size of a black hole with respect to its mass and it can be determined from the escape velocity equation. Schwarzschild radius is the magnitude of radius inwich the mass of an spherical celestial body should be concentrated to the speed of light corresponds to the escape velocity. The Schwarzschild radius [11] is defined as

$$r_S = \frac{2GM}{c^2}, \quad (24)$$

where c is the speed of light in vacuum (equals to 299,792.458 km/s^2) and M is the mass of the black hole. Considering the mass of Sun, for instance, the corespondent Schwarzschild radius is approximately of 2.95 km.

The Schwarzschild radius is also linked to gravitational colapse as the black holes formation, but hopoteticaly, a body can be orbiting a black hole at some given distance far enough of the event horizon. Thus, from expression (24), correspondence between escape velocity in Schwarzschild radius and escape velocity of any celestial body described in expression (3), is commonly defided as

$$v_e^2 = \frac{2GM}{r_e} \Rightarrow c^2 = \frac{2GM}{r_S}, \quad (25)$$

where escape velocity corresponds to the speed of light and escape radius corresponds to the Schwarzschild radius.

Although, this defined escape velocity equation for a given black hole is based by comparison on the classical equation (3) and not the relativistic, it is still valid [11].

According to this given correspondence in equivalence (25), extending expression (22) for a circular orbit around of a black hole, the equivalent expression is given by

$$c^2 r_S = v_o^2 (r_o + r_o) = 2v_o^2 r_o, \tag{26}$$

and finding out for speed of light from equivalence (26), yields

$$c = \sqrt{\frac{2v_o^2 r_o}{r_S}}. \tag{27}$$

Considering the Schwarzschild radius for a celestial body with a mass like one of the Sun, in the Solar System case Table I is built by substituting correspondent planetary data [5, 6] in equivalent expression (27), where speed of light amount is obtained respectively, then confirming the applicability of this equivalence between the related velocities and their respective radius in such an expression. Deviation is due to the accuracy of the data.

TABLE I. Data of planets with calculated speed of light.

Planet	*Mean Sun-planet distance (r_o) (10^6 km)*	*Mean orbital velocity (v_o) (km/sec)*	*Calculated equivalence with speed of light (km/sec)*	*Deviation (%)*
Mercury	57.91	47.87	299,767.89	0.008
Venus	108.208	35.02	299,785.05	0.002
Earth	149.60	29.80	299,934.88	0.047
Mars	227.963	21.97	299,795.77	0.001
Jupiter	778.60	12.44	300,108.08	0.105
Saturn	1,433.50	0.09	301,903.04	0.704
Uranus	2,872.50	6.29	300,346.13	0.184
Neptune	4,495.10	5.37	299,581.11	0.070

V. CONCLUSIONS

By reviewing both, the classical escape velocity and orbital velocity of a body that orbits around of a central force, we can derive some equivalent expressions between both velocities. Specifically, one of these equivalent expressions indicates that square of escape velocity multiplied by the escape radius is equivalent to the square of orbital velocity multiplied by the orbital diameter, which we can extend to the black hole case in order to relate both, the escape velocity of a black hole and the orbital velocity of a body in a circular orbit around of a black hole.

Although some equivalent expressions have been derived from algebraic equivalences and sometimes they have not an intrinsic physical referent or meaning, equivalences here derived can be applied in education to extend the explanation of the motion of the bodies under a gravitational force and to show the existing relation between the escape velocity and the orbital velocity, which comes from a common expression of conservation of total energy in a gravitational system.

Such equivalences can be also applied in education to derive the Newtonian expression for the Kepler's third law for both, elliptical or circular orbit, through to their relation between the velocity of a body and its position with respect to the central force. Extending the equivalences to the case of a central force as a black hole, it is possible to improve the understanding of the Kepler's laws in a gravitational system of two bodies and their relation with the Schwarzschild radius.

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REFERENCES

[1] Tan, A., *Theory of Orbital Motion* (World Scientific Publishing Co. Pte. Ltd., Singapore, 2008), pp. 26-44.

[2] Myers, R.L., *The Basics of Physics* (Greenwood Publishing Group, Inc., USA, 2006), pp. 52 – 58.

[3] Chang, H. and Hsiang, W., The epic journey from Kepler's laws to Newton's law of universal gravitation revisited, AJP, V1 (2008) pp. 7 – 17, [arXiv:0801.0308v1]

[4] Goldstein H., Classical Mechanics, 2nd ed. (Addison-Wesley Publishing Company, Inc. Reading, Massachusetts, 1980), pp. 91-103.

[5] Beatty, J. K., Petersen C. C. and Chaikin A., *The New Solar System*, 4th ed. (Cambridge University Press, Cambridge, 1990).

[6] Williams, D. R., *Planetary Fact Sheet – Metric*, (NASA, Houston, Texas, 2007). Visited on June 18th, 2010. http://nssdc.gsfc.nasa.gov/planetary/factsheet

[7] Raine, D. and Thomas, E., *Black Holes: An Introduction*, 2nd ed. (Imperial College Press, London, 2009) pp. 2 – 6.

[8] Kutner, M.L., *Astronomy* (Cambridge University Press, Cambridge, 2003) pp. 148 – 156.

[9] Laplace, P. S., *Exposition du Système du Monde*, 1st ed. (Cercle-Social, Paris, 1796).

[10] Schwarzschild, K., *Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie*, Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin, Klasse fur Mathematik, Physik, und Technik (Berlin, 1916), p. 424.

[11] Preti, G., *Schwarzschild Radius Before General Relativity: Why Does Michell-Laplace Argument Provide the Correct Answer?*, Foundations of Physics **39**, 1046-1054 (2009)