Simplified version of Curzon-Ahlborn engine with general heat transfer

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Abstract

A general model of the Curzon-Ahlborn type with general heat transfer law is discussed. A simplified version of this model is introduced. The effect of the ratio of heat transfer coefficients on the efficiency of the simplified version of Curzon-Ahlborn engine is presented. While the efficiency of the Curzon-Ahlborn engine at maximum power production is independent of the ration of heat transfer coefficients, the efficiency of the simplified version is not independent on the aforementioned ratio. The expression for the efficiency at maximum power production is derived in closed form. In order to highlight the differences visually and for the benefits of undergraduate students, power - efficiency curves are produced for both models. The validity of this simplified version model is discussed. Finally, simple derivation for the efficiency at maximum power production of the simplified model with n=1 is demonstrated.

Keywords: Curzon-Ahlborn model, temperature difference, general heat transfer law, thermal resistance, simplified version, power output, efficiency, Validity of the simplification.

Resumen

Se discute un modelo general del tipo de Curzon-Ahlborn con la ley general de transferencia de calor. Se introduce una versión simplificada de este modelo. Se presenta el efecto de la razón de los coeficientes de transferencia de calor en la eficiencia de la versión simplificada del motor de Curzon-Ahlborn. Si bien el rendimiento de la máquina Curzon-Ahlborn en la producción de potencia máxima es independiente de la razón de los coeficientes de transferencia de calor, la eficiencia de la versión simplificada no es independiente de la relación antes mencionada. La expresión de la eficiencia en la producción de potencia máxima se obtiene en forma cerrada. A fin de destacar las diferencias visuales y para beneficios de estudiantes de pregrado, las curvas de eficiencia de potencia se producen para ambos modelos. Se discute la validez de esta versión simplificada del modelo. Por último, se demuestra la derivación simple de la eficiencia en la producción de potencia máxima del modelo simplificado con n = 1.

Palabras clave: Modelo de Curzon-Ahlborn, diferencia de temperatura, ley general de transferencia de calor, Resistencia térmica, versión simplificada, potencia de salida, eficiencia, validación de simplificación.

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I. INTRODUCTION

The efficiency of heat engines has an upper bound, the Carnot efficiency [1]. This upper bound is achieved under the assumption of zero heat transfer rates, which results zero power output. Real heat engines run in finite time, thus constraining the operation to account for finite heat transfer rates. Curzon & Ahlborn [2] relaxed the conditions of Carnot engine to include finite times and finite heat transfer rates. Following these assumptions, the derived efficiency at maximum power operation is known as the Curzon-Ahlborn efficiency. Both efficiencies; the Carnot efficiency and the Curzon-Ahlborn efficiency are independent the ration of heat transfer coefficients at both boundaries of the heat engine and the heat reservoirs. Recently, Agrawal [3] analyzed the Curzon-Ahlborn model constraining the heat transfer rates at both sides of the engine to be the same. The resulted efficiency at maximum power operation was slightly different from the Curzon-Ahlborn efficiency. The present note extends the analysis to account for a general heat transfer law, following the ideas of Shaojun *et al.* [7]. This note also discusses the effect of heat conductance ratio on the efficiency for maximum power operation of the simplified model, compares the efficiencies of the two models and discusses the differences between the efficiencies. At the end, a simple derivation is introduced based on the Curzon-Ahlborn expression for the efficiency at maximum power operation. Section II presents the the Curzon-Ahlborn engine with a general heat transfer law, Section III focuses on the effect of the heat conduction ratio and reproduces the results of the simplified version of the Curzon-Ahlborn model, Section IV discusses the validity of the simplified version model and simple derivation is presented, Section V compares the power-efficiency curves for the case n=4, and finally Section VI reports the concluding remarks.

II. MODEL FORMULATION

Consider a heat engine working between two heat reservoirs with T_1 is the temperature of the hot reservoir and T_3 is the temperature of the cold reservoir. Due to thermal resistance at the hot side of the engine, it senses lower temperature T_h ($T_h < T_1$) and the cold side rejects heat to lower temperature T_c ($T_c > T_3$). The schematics of the engine are shown in Fig. 1a, and its T-S diagram is shown in Fig. 1b. The following equations are written for the endo-reversible heat engine.

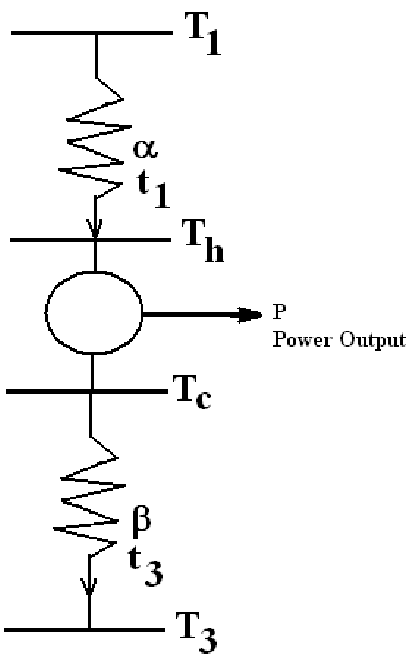

FIGURE 1a. Schematic diagram of the endo-reversible engine.

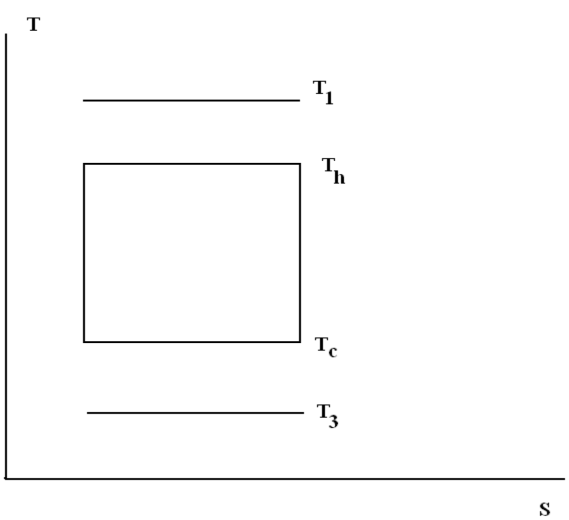

FIGURE 1b. T-S diagram of the endo-reversible heat engine.

The heat input to the heat engine Q_1 is given by:

$$Q_1 = \alpha x t_1. \tag{1}$$

Where α represents the heat conductance at the hot side, t_1 is the time spent at the hot side and x is the temperature difference at the hot side and is given by:

$$x = T_1^n - T_h^n. \tag{2}$$

Similarly, the heat output from the engine Q_3 is given by:

$$Q_3 = \beta y t_3. \tag{3}$$

Where β represents the heat conductance at the hot side, t_3 is the time spent at the hot side and y is the temperature difference at the cold side and is given by:

$$y = T_c^n - T_3^n. \tag{4}$$

The endo-reversibilty condition states that the entropy generation of a complete cycle should satisfy the following condition:

$$\frac{Q_1}{T_h} = \frac{Q_3}{T_C}. \tag{5}$$

The power output from the heat engine P is given by the difference between heat input to the heat engine and the heat rejected from the heat engine. The expression for power is given by:

$$P = \frac{Q_1 - Q_3}{t_0}. \tag{6}$$

Where t_0 is the cycle time of the heat engine, which is the sum of the two isothermal and the two adiabatic branches. The time spent at the adiabatic branches is assumed to be proportional to the time spent at the isothermal branches. Then, the cycle time is given by:

$$t_0 = \gamma(t_1 + t_3). \tag{7}$$

Where γ - 1 represents the fraction of the time spent at the adiabatic branches.

Finally, the efficiency η is defined as the ratio of work output and the heat input and it is given by the expression:

$$\eta = 1 - \frac{Q_3}{Q_1}. \tag{8}$$

Equations (1) – (8) serve as the basic equations of the Curzon-Ahlborn model and its simplified version with the constraint of equal heat transfer rates (which basically are the product of the heat conductance by the temperature difference).

In the following subsection, explicit expressions for the power output are derived as a function of the efficiency of the heat engine and the times spent at the hot and cold sides of the heat engine. This kind of expression is useful for plotting the characteristic curves of the heat engine, represented by power-efficiency plots.

A. Curzon-Ahlborn engine with general heat transfer law

In this section the expressions for the heat input and the output power are derived for the Curzon-Ahlborn engine. These expressions are useful for producing power-efficiency plots. After manipulating equations (1) – (8) (see appendix a for detailed derivation), the heat input to the engine Q_{1CA} is given by:

$$Q_{1CA}=\frac{(1-\eta)^n T_1^n - T_3^n}{(1-\eta)\left(\frac{(1-\eta)^{n-1}}{\alpha t_1}+\frac{1}{\beta t_3}\right)}. \qquad (9)$$

Manipulating equations (6) – (8) gives the expression for power output of the Curzon-Ahlborn engineP_{CA}:

$$P_{CA}=\frac{\alpha}{\gamma}\frac{\eta((1-\eta)^n T_1^n - T_3^n)}{(1-\eta)\left((1-\eta)^{\frac{(n-1)}{2}}+1/\sqrt{k}\right)^2}. \qquad (10)$$

Here k is the ratio between α and β.

It is important to note that using these coordinates (*i.e.*, time spent at each branch and efficiency) decouples the dependencies on time and efficiencies (after performing optimization with respect to time). This simplifies the optimization process. Equation (10) is written after performing the first optimization with respect to time division.

Maximizing the power output for the case n=1, with respect to efficiency led to the well-known Curzon-Ahlborn efficiency:

$$\eta_{CA}=1-\sqrt{\tau}=1-\sqrt{1-\eta_c}. \qquad (11)$$

Here τ is the ratio between T_3 and T_1 ($\tau = {T_3}/{T_1}$) and η_c is the Carnot efficiency.

III. SIMPLIFIED VERSION MODEL WITH DIFFERENT HEAT CONDUCTANCES

Repeating the algebraic manipulation for the simplified version of the Curzon-Ahlborn model as described in [3] in detail considering general heat transfer model, assuming x=y but different heat conductance coefficients, led to the following expressions of heat input and power output. The expression of the heat input to the simplified version of the engine with different values of heat conductances Q_{1VK} is given by:

$$Q_{1SVK}=\frac{\alpha t_0}{\gamma}\frac{((1-\eta)^n T_1^n - T_3^n)}{(1+\frac{(1-\eta)}{k})\left(1+(1-\eta)^n\right)}. \qquad (12)$$

The power output for the simplified version of the engine with different heat conductances P_{SVK} is given by:

$$P_{SVK}=\frac{\alpha}{\gamma}\frac{\eta((1-\eta)^n T_1^n - T_3^n)}{(1+\frac{(1-\eta)}{k})\left(1+(1-\eta)^n\right)}. \qquad (13)$$

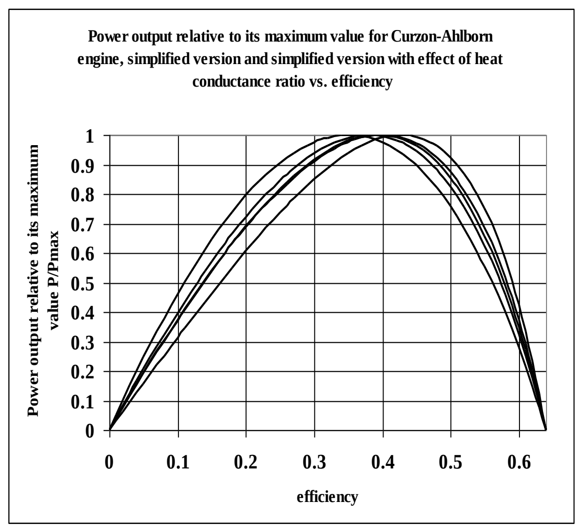

FIGURE 2a. Power output relative to its maximum value as a function of efficiency for the Curzon-Ahlborn model and the simplified model. For the simplified model, plots are produced for different values of the ration of heat transfer coefficients: κ=0, 1, 2, 1000.

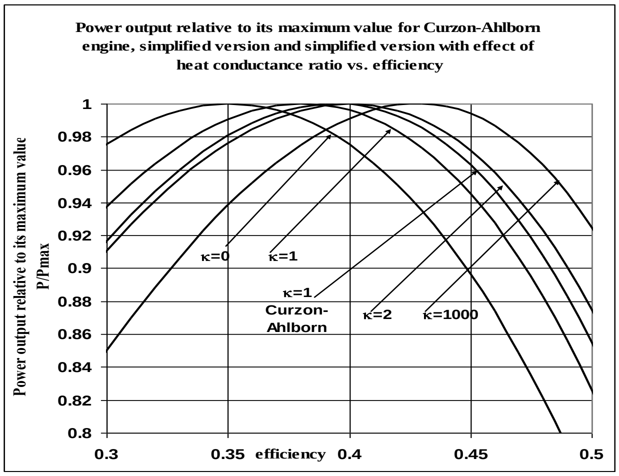

FIGURE 2b. Enlargement of Fig. 2a in order to increase clarity.

Fig. 2a shows the plot of the power output relative to its maximum value of the Curzon-Ahlborn model (with n=1) and of the simplified version for different values of the heat conductance ratio. It is evident from these plots that the efficiency at maximum power point shifts around the Curzon-Ahlborn efficiency. Fig. 2b is enlargement of Fig. 2a near the maxima in order to identify the curves.

The expressions for the simplified version of the Curzon-Ahlborn model could be easily produced by replacing the parameter k by one (k=1).

It is important here to note that the time division is not independent of efficiency due to the assumption of equal temperature differences at both side of the simplified version of the heat engine. Differentiating the power expression given by equation (13) with respect to efficiency gives the efficiency at maximum power production,

$$\eta_{MAX,SVK} = \frac{\eta_c}{1+\sqrt{1-\frac{\eta_c(1+3k-k\eta_c)}{2(k+1)}}}. \qquad (14)$$

From equation (14) one can note that the efficiency of the simplified model with different heat conductance values is not independent of the ratio k.

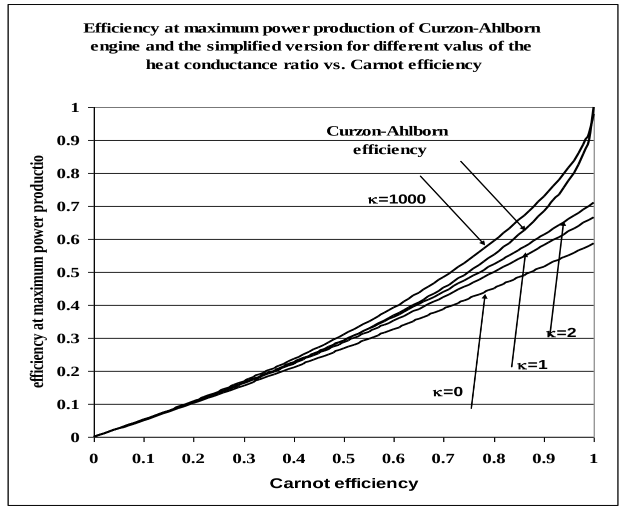

FIGURE 3. Curzon-Ahlborn efficiency and the efficiency at maximum power production of the simplified version vs. the Carnot efficiency.

Fig. 3 shows the maximum efficiency of the Curzon-Ahlborn (with n=1) and the efficiency at maximum power production for the simplified version as a function of the Carnot efficiency for different values of heat conductance ratio. It is clear from the plot that the differences are not negligible at the higher range of Carnot efficiency.

Fig. 4 shows the efficiency at maximum power output of the simplified model (with n=1) as a function of the thermal conductance ratio and for Carnot efficiency of 64%. For the West Thurrock Coal Fired Steam Plant, the efficiency increases from 35% up to 43%.

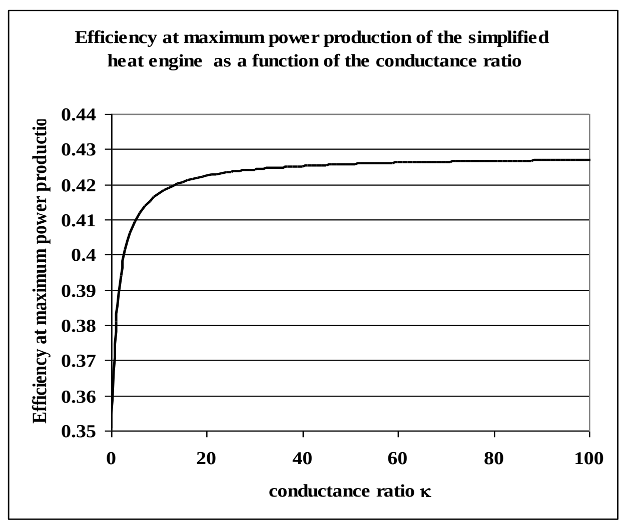

FIGURE 4. the efficiency at maximum power production of the simplified version of the Curzon-Ahlborn engine.

To reproduce the efficiency at maximum power production as reported in [3] (written in terms of the Carnot efficiency), we substitute k=1 to get:

$$\eta_{MAX,SV} = \frac{\eta_c}{1+\sqrt{1-\frac{\eta_c(4-\eta_c)}{4}}} = \frac{\eta_c}{2-\frac{\eta_c}{2}}. \quad (15)$$

For a case where $\eta_C << 1$ the efficiency at maximum power production is given by:

$$\eta_{MAX,SV} = \frac{\eta_c}{2}. \quad (16)$$

The operative efficiency of the heat engine will range from the efficiency at maximum power production to the Carnot efficiency, thus the operative efficiency could be approximated as the mean value of both efficiencies and it would be given by:

$$\eta_{opr} = \frac{\eta_C + \frac{\eta_C}{2}}{2} = \frac{3}{4}\eta_C. \quad (17)$$

In the next section, we derive the last expression in a much simpler way based on the Curzon-Ahlborn efficiency.

IV. VALIDITY OF THE SIMPLIFIED MODEL

In this section, the validity of the simplified model is discussed. The main question: what is the reason of the good agreement between the efficiency of the simplified model (with n=1) and the Curzon-Ahlborn efficiency? In order to answer this question, we compared the differences between x and y values of both models. By adopting the results given in [3] the difference Δx or equivalently Δy *is* given by:

$$\Delta x = x_{CA} - x_{SV} = \left(\frac{\sqrt{T_1}-\sqrt{T_3}}{2}\right)^2. \quad (18)$$

This expression could be written in a dimensionless form by dividing by T_1. The normalized expression is given by:

$$\Delta x^* = \frac{\Delta x}{T_1} = \left(\frac{1-\sqrt{\tau}}{2}\right)^2. \quad (19)$$

The results of substituting numerical values for the three cases given in [3] shows that the value of Δx is less than 5%. This small variation of the result allows expansion of the Curzon-Ahlborn efficiency to get different kinds of approximation. In order to demonstrate the differences we perform the calculation for all the cases given in [3]:

1. West Thurrock (U.K.) Coal Fired Steam Plant [4]

$$T_1 = 838K, \quad T_3 = 298K, \quad \Delta x = 34K, \quad \Delta x^*\% = 4\%. \quad (20)$$

2. CANDU (Canada) PHW Nuclear Reactor [5]

$$T_1 = 573K, \quad T_3 = 298K, \quad \Delta x = 11K, \quad \Delta x^*\% = 1.9\%. \quad (21)$$

3. Larderello (Italy) Geothermal Steam Plant[6]

$$T_1 = 523K, \quad T_3 = 353K, \quad \Delta x = 4K, \quad \Delta x^*\% = 0.8\%. \quad (22)$$

If we approximate the Curzon-Ahlborn efficiency we can get the same result of the simplified version model as follows:

1) Write the expression in terms of Carnot efficiency. The result is given by equation (10).

2) Multiply and divide by the conjugate of the expression given by equation (10).

3) Approximate the square root up to the first term of Taylor series.

Performing the steps one by one by using mathematical symbols is shown by:

$$\eta_{CA} = 1-\sqrt{1-\eta_C} = \left(1-\sqrt{1-\eta_C}\right)\frac{1+\sqrt{1-\eta_C}}{1+\sqrt{1-\eta_C}} = \frac{\eta_C}{1+\sqrt{1-\eta_C}}$$

$$\approx \frac{\eta_C}{1+1-\frac{\eta_C}{2}} = \frac{\eta_C}{2-\frac{\eta_C}{2}}. \quad (23)$$

The result is exactly the efficiency at maximum power production of the simplified model as given by equation (15).

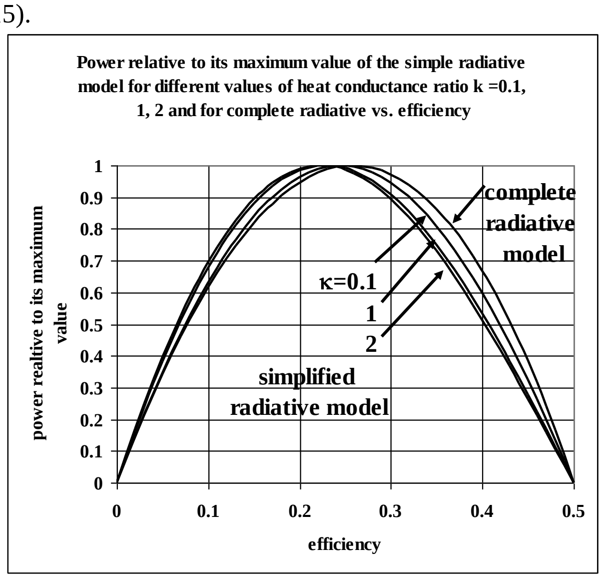

FIGURE 5a. Power relative to its maximum value for the case n=4 and for τ=0.5. The plot compares between the exact model when k=1 and the simplified model for k=0.1, 1 and 2.

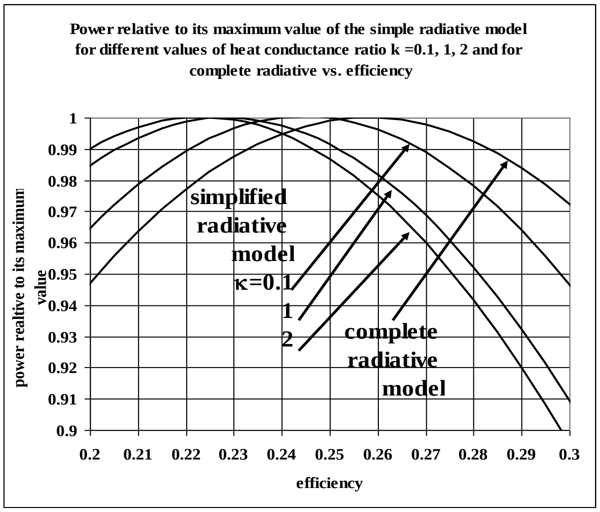

FIGURE 5b. Enlargement of fig. 5a to increase clarity.

V. POWER-EFFICIENCY CURVES FOR *n*=4

In this section the power - efficiency curves for the case n=4 are compared. Plot 5.a and 5.b show the relative power output of the Curzon-Ahlborn model with n=4 for the exact model and for the simplified model with different values of κ (κ=0.1, 1, 2). It is obvious from the plots that the efficiencies at maximum power production are slightly different. That means with much less effort it is possible to easily estimate the performance of the heat engine based.

VI. CONCLUDING REMARKS

The salient features of the present work can be summarized as follows:

- A simplified Curzon-Ahlborn model with general heat transfer law is introduced.
- The effect of heat conductance ratio on the maximum efficiency of the simplified version of the Curzon-Ahlborn is discussed and it was found that the value of the efficiency varies significantly as shown in figures 3 and 4.
- The endo-reversible power output can be expressed in terms of efficiency. It turns out to be that the power output expression is very simple as given by equation (9).
- The temperature difference x between the values of the Curzon-Ahlborn engine and its simplified version explains the small differences while comparing the efficiencies.
- The simplified version of the Curzon-Ahlborn (for n=4, see Fig. 5a and 5b) produces values of efficiency at maximum power production, not far from the results of the exact model.
- Another simple derivation was presented, by using simple and very useful a mathematical trick. In fact, starting from the Curzon-Ahlborn efficiency, we derived the efficiency at maximum power production of the simplified version.
- The present derivation is quite simple and can be easily reproduced by the students.

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Appendix A

From Eq. (1) and Eq. (3) we get:

$$\frac{Q_3}{Q_1}=\frac{\beta y t_3}{\alpha x t_1}. \qquad (A1)$$

Rearranging Eq. (2) gives:

$$T_h^n = T_1^n - x. \qquad (A2)$$

Rearranging Eq. (4) gives:

$$T_c^n = T_3^n + y. \qquad (A3)$$

Rearranging Eq. (8) and Eq. (5) give:

$$\frac{Q_3}{Q_1}=\frac{T_c}{T_h}=1-\eta. \qquad (A4)$$

Squaring Eq. (A4) and substituting the relations given in Eq. (A2) and Eq. (A3) leads to:

$$\frac{T_c^n}{T_h^n}=(1-\eta)^n=\frac{T_3^n+y}{T_1^n-x}. \qquad (A5)$$

Rearranging Eq. (A5) gives:

$$\left(1-\eta\right)^{n} T_{1}^{n} - \left(1-\eta\right)^{n} x = T_{3}^{n} + y. \qquad (A6)$$

Substitution Eq. (A1) in Eq. (a4) gives:

$$1-\eta = \frac{\beta t_3 y}{\alpha t_1 x}. \qquad (A7)$$

Substituting y from Eq. (A7) in Eq. (A6) and solving for *x*, one gets:

$$x = \frac{\left(1-\eta\right)^{n} T_{1}^{n} - T_{n}}{\left(1-\eta\right)^{n} + \left(1-\eta\right)\dfrac{\beta t_1}{\beta_3}}. \qquad (A8)$$

Now substituting *x* in Eq. (1) and rearranging terms, gives Eq. (9).

In order to equation (10) we multiply the expression of the heat input Q_1 with the efficiency and divide by cycle time. Differentiating the power expression with respect to the time spent at the hot side (under the assumption of total cycle time), and equating the derivative to zero, leads to the aforementioned expression of power output.