Special Relativity and textbook exercises

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Abstract
A textbook exercise about a block pulled on a rough surface by a conservative force and against a dissipative frictional force is described using a 4-vector relativistic thermodynamics approach, taking into account thermal effects produced in diathermal contact with a heat reservoir. The conservative force is described using a relativistic electromagnetic formalism and the whole exercise is solved in a Lorentz covariant form, first in ‘privileged frame’ $S_\infty$ in which both surface and heat reservoir are at rest, and after that in frame $S_A$, in standard configuration with respect to frame $S_\infty$. This exercise could be of interest to undergraduate students with an interest in Special Relativity and Thermodynamics.

Keywords: Special Relativity, four-vectors, relativistic thermodynamics first law.

I. INTRODUCTION

Introductory physics textbooks consider exercises in which a solid block moves on a rough surface with a friction force applied to it (Fig. 1) ([1], p. 198). A block $Z$, with mass $m$, is accelerated from rest along a table by a force $F$ (all forces are one-dimensional and vectorial notation is dropped) applied to it and against a sliding frictional force $f$. It is found by experiment that a portion of the table along which the block slides becomes warmer ([2], pp. 153-154). The thermal effects of friction forces are rarely formalized in university textbooks ([3], pp. 204-207).

It is always interesting to ask for the description of the same process from the point of view of an observer that moves with constant speed with respect to the ‘privileged frame’ $S_\infty$ (lab frame) [4]. Galileo’s Principle of Relativity is rarely applied to textbook exercises that include thermal effects.

On the other hand, university physics textbooks do not solve this kind of exercise either using concepts of special relativity ([5], pp. 359, 361) or Minkowski’s 4-vector formalism [6]. This absence of special relativity and 4-vector formalism in the resolution of exercises in university physics textbooks is a pedagogical shortcoming [7].
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When an exercise is solved in the relativistic Einstein-Minkowski 4-vector formalism, the solution is Lorentz invariant, in contrast with the classical solution, which is not [8]. Moreover, the Newton-Galileo solution is obtained in the limit in which \( c \to \infty \).

In contrast to what happens in university textbooks, the special theory of relativity continues to arouse great interest among both teachers [9] and students, as is evidenced by the important number of articles on this topic [10] published during the past few years in journals devoted to undergraduate physics education [11].

![Figure 2: Relativistic description of the situation in Fig. 1. Clocks at rest in \( S_\infty \) indicates its time \( t \). Body \( Z \) proper time \( \tau \) is measured by a clock attached to it [11]. Thermal energy is characterized as photons with frequency \( \nu \) absorbed by the heat reservoir \( B_\infty \).](image)

**II. RELATIVISTIC THERMODYNAMICS**

**FIRST LAW**

Consider the relativistic description of the process given in Fig. 2. A rigid (see Appendix) body \( Z \) with constant inertia \( M_Z \) (see Appendix) which moves with initial velocity \( v_i \) in frame \( S_\infty \), has linear momentum \( p_i = \gamma(v_i)M_Zv_i \) and total energy \( E_i = \gamma(v_i)M_Zc^2 \) [12], where \( \gamma(v) = (1 - v^2/c^2)^{-1/2} \), and its initial 4-vector [13] energy function [14] \( U_i^\mu \) is (for typographical reasons, a contravariant column 4-vector is expressed as a row 4-vector, maintaining its contravariant Greek index):

\[
U_i^\mu = \{c\gamma(v_i)M_Zv_i, 0, 0, \gamma(v_i)M_Zc^2\}. 
\]  

(1)

For a finite process, during which different forces are applied to \( Z \) during a time interval \( t_0 \), it reaches a final velocity \( v_f \), and its 4-vector final energy function \( U_f^\mu \) is:

\[
U_f^\mu = \{c\gamma(v_f)M_Zv_f, 0, 0, \gamma(v_f)M_Zc^2\}. 
\]  

(2)

For a conservative force \( F = qE \) with associated 4-vector displacement \( L = \{L, 0, 0, ct_0\} \), \( L \) being the displacement of its centre-of-mas, the 4-vector work \( W_f^\mu \) is (see Appendix):

\[
W_f^\mu = \{cqEt_0, 0, 0, qEL\}. 
\]  

(3)

This 4-vector \( W_f^\mu \) is expressed as the (minus) increment in energy function of a work reservoir (the battery that provides the electric charge on capacitor and the electric field), with \( W_f^\mu = -\Delta U_f^\mu \) [15].

In a process in which body \( Z \) slides on the surface of the quasi-infinite inertia system \( B_\infty \) [4], the force \( f \) is described using two phenomenological rules ([16], pp. 6-7): Amonton’s rule, where the frictional force \( f \) is proportional to the normal \( N \) holding the two surfaces, and it is independent of the area of the surfaces in contact, and Coulomb’s rule, where force \( f \) is independent of the sliding velocity ([17], pp. 884-885). We assume that force \( f \) is described in special relativity using these rules, with: \( f = -\mu mg \), where \( N = M_Zg \). A friction force has no associated displacement in frame \( S_\infty \), in which \( B_\infty \) remains at rest ([18], p. 616), its ‘product force-displacement’ [19] is null and it is a zero-work force [20]. The 4-vector work \( W_f^\mu \) for \( f \) is:

\[
W_f^\mu = \{-c\mu M_Zgt_0, 0, 0, 0\}. 
\]  

(4)

The 4-vector \( W_f^\mu \) cannot be expressed as the increment in energy function of a work reservoir.

For a solid body, forces are simultaneously applied to it in frame \( S_\infty \) during time interval \( t_0 \) [21]. The 4-vector total work ([22], p. 84): \( W^\mu = W_f^\mu + W_r^\mu \) is:

\[
W^\mu = \{c(qE - \mu M_Zg)t_0, 0, 0, qEL\}. 
\]  

(5)

When a friction force acts during a certain mechanical process on a rigid body, like the process described in Fig. 2, mechanical energy is dissipated as heat. A complete description of such a process includes a relativistic description of heat [23], and its corresponding 4-vector thermal energy \( Q^\mu \).

Energy interchanged as heat is formed by photons [24] with total zero linear momentum (thermal photons) (see Appendix). A photon with frequency \( \nu \) has energy \( E_p = h\nu \). In the thermal photons monochromatic approximation [25] it is assumed that every interchanged photon has the same frequency. During a friction process, total energy \( E_p = Nh \) is interchanged as thermal energy, where \( N \) is the number of photons transferred. We take heat \( Q \) to be positive when heat is transferred to body \( Z \) and negative when the transfers are out of the body. With \( Q = -Nh \), the 4-vector heat \( Q^\mu \) is given in frame \( S_\infty \) as [8]:

\[
Q^\mu = \{0, 0, 0, -Nh\}. 
\]  

(6)
The inertia of this ensemble of photons [26] is $M_r = (N\hbar v)c^2$ ([27], p. 232). The 4-vector $Q^\mu$ can be expressed as the (minus) increment in energy function of a heat reservoir (the floor $B_o$) at temperature, $T$, with $Q^\mu = -\Delta U^\mu_r$ [15].

The Relativistic Thermodynamics First Law in frame $S_\infty$, for the process under consideration is given by [8]:

$$\Delta U^\mu_r + \Delta U^\mu_F + \Delta U^\mu_t = W^\mu,$$  \hspace{1cm} (7)

This equation emphasizes the role played by the heat reservoir $B_o$ as a thermal radiation sink, and the role as work reservoir of the battery that charges the capacitor C, which provides the work done on Z ([28], p. 290).

From Eq. (8) and the previously obtained 4-vectors, one obtains the equations:

$$\gamma(v_f)M_zv_f - \gamma(v_i)M_zv_i = (\gamma c - \mu M_z g) t_0,$$  \hspace{1cm} (9)

$$\gamma(v_f)M_zc^2 - \gamma(v_i)M_zc^2 = \gamma c L - Nh_2.$$  \hspace{1cm} (10)

Eq. (9) is the relativistic impulse-momentum equation (relativistic Newton’s second law) for the process under consideration and Eq. (10) is the corresponding energy equation (relativistic energy conservation law). Taking into account that ([29], p. 37):

$$\frac{d[\gamma(v)c^2]}{d[\gamma(v)v]} = v = \frac{dx}{dt},$$

for a finite process with constant force $F$, applied during time interval $t_0$, and constant inertia $M_z$, with the body’s centre-of-mass displacement $L$:

$$M_z \Delta \left[\frac{\gamma(v)c^2}{\gamma(v)v}\right] = FL,$$

$$M_z \Delta \left[\frac{\gamma(v)c^2}{\gamma(v)v}\right] = FL.$$  \hspace{1cm} (11)

When the impulse-momentum Eq. (9) is fulfilled then the relativistic centre-of-mass equation is fulfilled too, with:

$$\gamma(v_f)M_zc^2 - \gamma(v_i)M_zc^2 = (\gamma c - \mu M_z g) L.$$  \hspace{1cm} (11)

This relativistic centre-of-mass equation is expressed as $\Delta K_z = F \Delta x_m$, where $K_z$ is the kinetic energy, $F$ is the resultant force applied on $Z$ and $\Delta x_m$ is its centre-of-mass displacement [30]. Comparing the energy equation Eq. (10) and the centre-of-mass equation, Eq. (11), one obtains the thermal energy equation:

$$Nh_2 = \mu M_z g L,$$  \hspace{1cm} (12)

with $\Delta U_T = \mu M_z g L$.

A. Description in frame $S_A$

For an observer in frame reference $S_A$ in standard configuration with respect to $S_\infty$ [31], with velocity $\vec{v} = (V, 0, 0)$ with respect to frame $S_\infty$, the Lorentz transformation matrix for the standard configuration is ([6], p. 236):

$$L^\mu_r(V) = \begin{bmatrix}
\gamma(v) & 0 & 0 & -\beta(v)\gamma(v) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\beta(v)\gamma(v) & 0 & 0 & \gamma(v)
\end{bmatrix},$$  \hspace{1cm} (13)

with $\beta(v) = V/c$ and $\gamma(v) = [1 - \beta^2(v)]^{-1/2}$. When an exercise is solved in the frame $S_r$ using 4-vector equations, the corresponding magnitudes (velocity, force, frequency, etc.) in $S_A$ are obtained through relativistic transformation (relativistic transformation of velocity, relativistic force transformation, Doppler effect, etc.) and the corresponding 4-vectors in $S_A$ are obtained through the Lorentz transformation [32]. For example, from 4-vector initial velocity $v_i^\mu$:

$$v_i^\mu = \{\gamma(v_i)v_i, 0, 0, \gamma(v_i)c\},$$

the corresponding 4-vector $v_{iA}^\mu$,

$$v_{iA}^\mu = L_i^\mu_r(V)v_i^\nu = \{\gamma(v_{iA})v_i, 0, 0, \gamma(v_{iA})c\},$$

implies that:

$$\gamma(v_{iA})v_{iA} = \gamma(v)\gamma(V)(v - V),$$  \hspace{1cm} (14)

$$\gamma(v_{iA}) = \gamma(v)\gamma(V)\left(1 - vV/c^2\right),$$  \hspace{1cm} (15)

and

$$v_{iA} = \frac{v_i - V}{1 - vV/c^2}.$$  \hspace{1cm}

Any other magnitude in $S_A$ is obtained in a similar way [8].

The Lorentz transformation immediately provides the correct description in $S_A$ for the process previously described in $S_\infty$. For instance, with

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From Eqs. (9, 10, 11), in the limit $c \to \infty$ [with $\lim_{c \to \infty} \gamma(v) v \to v$ and $\lim_{c \to \infty} \gamma(v) \to 1$], $e^2 \to e^2 / 2$, the corresponding classical description for the considered process $(M_Z \equiv m)$ is:

$$mv_f - mv_i = (qe - \mu mg) t_0;$$  \hspace{1cm} (21)

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = qeL + Q;$$  \hspace{1cm} (22)

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = (qe - \mu mg)L;$$  \hspace{1cm} (23)

$$Q = -\mu mg L.$$  \hspace{1cm} (24)

Eq. (21) (impulse-momentum equation) is the Newton’s second law applied to $Z$; Eq. (22) is the corresponding first law of thermodynamics ([1], p. 194):

$$\Delta K_{cm} + \Delta U = W + Q,$$

applied to the process [35, 36], with $\Delta U = 0$ because $Z$ temperature is constant – only the conservative force $F = qE$ applied to $Z$ does work $-$; Eq. (23) is the centre-of-mass equation which can be obtained considering that ([37], p. 1063):

$$\frac{d[v^2 / 2]}{dt} = v \frac{dx}{dr},$$

with constant mass $m$ and force $F$ and from Eq. (21) one obtains that:

$$m\Delta[v^2 / 2] = F \Delta x,$$

for a finite process ([5], pp. 359, 361); Eq. (23) is not a work-energy theorem ([30], p. 506); and Eq. (24), obtained by comparison between Eq. (22) (energy equation) and Eq. (23) (centre-of-mass equation), completes the energy balance for the process, showing that the interchanged thermal energy is the difference (mechanical energy dissipated) between the work done on $Z$ and $Z$ increment in kinetic energy. An equivalent classical description in $S_\Lambda$, from Eqs. (17, 18) in the approximation $c \to \infty$, can be obtained.

Although the 4-vector relativistic thermodynamics first law formalism seems more complicated than the classical Newton’s second law and thermodynamics first law approach, it presents some pedagogical advantages: (i) The definition of the 4-vector energy function $U^\mu$ – the corresponding relativistic internal energy [38] – must be provided, based on the Einstein’s equation interpretation for an extended body; (ii) For a conservative force $F$, its 4-vector work $W_F^\mu$ can be obtained from the interaction of the body with a work reservoir; (iii) For a non-conservative force $f$ [39] its ‘force-displacement product’ is zero (it does not do work) and its 4-vector has a null temporal
component; (iv) Heat can be characterised in frame $S_\alpha$ as a 4-vector $Q^\mu$ associated to an ensemble of photons with null total momentum [40]; (v) In frame $S_\alpha$ equations, expressed with 4-vectors, are Lorentz invariant and the description of a process in frame $S_\Lambda$ is obtained using the Lorentz transformation. Writing a physics law in Minkowski’s 4-vector notation, integrating space and time, highlights its invariance, simplifies the formalism and favours calculations. (vi) The relativistic impulse-momentum equation (Newton’s second law) and the relativistic energy equation (thermodynamics first law) are simultaneously applied to the process under consideration using the relativistic thermodynamics first law; (vii) The centre-of-mass equation is obtained integrating the corresponding impulse-momentum equation, showing that it is not an energy equation. (viii) Classical physics exercises can be solved using a fully Lorentz covariant formalism that includes mechanics, thermodynamics and electromagnetism 4-vector magnitudes.

APPENDIX

1. Energy function. Body $Z$ is a solid crystal composed by $N_A$ atoms of the atomic element $\alpha A$. Body $Z$ energy function $U(T)$ at absolute temperature $T$ is given by [8]:

$$U(T) = N_A u_0 - \bar{U} + \Delta U(T);$$

$$u_0 = N(m_p + m_e + m_n)c^2 + (Z - N)m_e c^2,$$

$$\bar{U} = \bar{U}_N + \bar{U}_A + \bar{U}_c,$$

$$\Delta U(T) = N_A \int_0^T c_p(T) dT$$

where $m_p$, $m_e$, and $m_n$ are proton, neutron and electron mass, respectively, and $\bar{U}$ is (negative) body $Z$ formation energy, or energy released when the body is formed from its elementary particles, with $\bar{U}_N$ as energy nucleus formation (defect of mass), $\bar{U}_A$ atom energy formation and $\bar{U}_c$ crystal energy formation (zero-point energy is assumed to be null [41]), respectively ([42], pp. 489-491), and where $c_p$ is solid capacity at constant pressure per atom (assuming expansion coefficient $\alpha = 0$).

The Inertia of Energy Principle [43] allows us to obtain the body’s inertia $M(T)$, at temperature $T$, as [44]:

$$M(T) = U(T)c^2.$$  

This equation (the Einstein Eq. [45]) relates two concepts, function energy and inertia, classically apart. An energy function increasing on a body correspondingly increases the body’s inertia: A block at a high temperature has more inertia than the same block at a lower temperature. Assuming an isothermal process for body $Z$ during the process, the body’s inertia —although temperature dependent— remains constant, with $M(T) = M_Z$.

2. Solid body in Relativity. In the Special Theory of

Relativity any perturbation on a body constituent travels with finite velocity. No body can be perfectly rigid in Relativity and it deforms under the action of a force applied to it ([46], p. 103). A deformation effect is transmitted to the rest of the body with finite velocity, $\nu$ s, the velocity of sound in the body’s material.

Let $b$ be a characteristic linear dimension of a rigid body $Z$. If the time interval $\Delta t = b/v$, time delayed for a sound wave to travel along the body — is orders of magnitude smaller than, for example, the time interval $\Delta t$ during which forces are applied to body $Z$, then, and under the action of a moderate force, it can be considered that $Z$ behaves as a rigid solid even from a relativistic point of view. We will consider a robust enough block so that its plastic distortion response to the applied force can be considered negligible.

The expression for a rigid body magnitude —linear momentum, total energy, kinetic energy, etc.— is obtained by changing the mass $m$ in the corresponding expression for a point particle for the inertia $M$ of the body (Principle of Similitude [8]). For instance, for a rigid body moving with velocity $v$ its kinetic energy is $k = [\gamma(v)-1]Mc^2$, which is obtained from the kinetic energy expression $k = [\gamma(v)-1]mc^2$ for a point particle with mass $m$.

3. Work. Body $Z$ (instantaneous) velocity $\vec{v} = (v,0,0)$ in $S_\alpha$ is characterized by the 4-vector velocity $\nu^\mu$, given by [47]:

$$\nu^\mu = \frac{dx^\mu}{d\tau} = \gamma(v)\{v,0,0,0\},$$

where $dx^\mu = \{dx,0,0,0\} \ dr$ is the 4-vector infinitesimal displacement and $\tau$ is body $Z$ proper time, with $d\tau = \gamma^2(v)$, where $\gamma$ is the corresponding time interval measured in frame $S_\alpha$. The 4-vector Minkowski force $F^\mu$ is given by Eq. ([16], p. 280)

$$F^\mu = \{\gamma(v)qe,0,0,0\},$$

and its corresponding infinitesimal 4-vector work $dW_f^\mu$ is [45]:

$$dW_f^\mu = \{eqxd\tau,0,0,0\}.$$  

The 4-vector Minkowski force $F^\mu$ is obtained by deriving the 4-vector work $dW_f^\mu$ with respect to $Z$ proper time $d\tau$:

$$F^\mu = c^{-1} \frac{dW_f^\mu}{d\tau},$$

This obtention of $F^\mu$ shows that $dW_f^\mu$ is itself a 4-vector.

For a finite process (with constant $F$):

$$W_f = \{eqx,t,0,0\}.$$  

In contact with $B_\alpha$, the friction force $f$ does not do work, its velocity is null, its 4-vector $J^\mu$ is given by:

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\[ f^\mu = \{-\mu M_Z g, 0, 0, 0\}, \]

and its 4-vector work \( W^\mu \) is:

\[ W^\mu = \{-\mu M_Z g t, 0, 0, 0\}. \]

4. Thermal energy. A photon \( j \) with frequency \( \nu \) has energy \( e_j = h \nu \), linear momentum \( p_j = \frac{(h \nu)c}{c} u_j \) (one-dimensional), where \( u_j \) is its direction \( \pm x \) and 4-vector energy function:

\[ u^\nu_j = \left\{ \frac{\pm h \nu}{c}, 0, 0, \nu \right\}. \]

For an ensemble of \( N \) photons which move chaotically in \( \pm x \) direction in a thermal reservoir, its total linear momentum is null [48], \( p_\nu = \sum (h \nu c) u_j = 0 \), its total energy is \( E_\nu = \sum h \nu |u_j| = Nh\nu \) and its 4-vector energy function is [45]:

\[ U^\nu = \{0, 0, 0, Nh\nu\}. \]

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