Anti-isospectral transformation on *PT*-symmetric Quantum Mechanics

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Abstract

We prescribe a transformation which act as parity, imaginary shift and anti-isospectral transformation. We discuss the effect of this transformation on different kind of potentials.

Keywords: *PT*-symmetry, Anti-isospectral, Pseudo-Hermitian.

Resumen

Prescribimos una transformación que actúa como la paridad, el cambio y la transformación de imaginarios anti-isospectral. Discutimos el efecto de esta transformación en diferentes tipos de potenciales.

Palabras clave: *PY*-simetría, Anti-isospectral, Pseudo-Hermite.

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I. INTRODUCTION

The Hermiticity of the Hamiltonian had been accepted as the necessary condition for the real spectrum since 1998 [1]. Few years ago, a more physical alternative axiom called PT-symmetry has been investigated [1]. The condition $H = H^{\dagger}$ is being replaced by weaker and more physical requirement $PTH = HPT$ where P denotes the parity ($PxP = -x$) and the anti-linear operator T mimics the time reflection ($TiT = -i$), one obtains new classes of complex Hamiltonian whose spectra are still real and positive. Mostafazadeh [2] in his very noteworthy work introduces the concept of pseudo-Hermiticity in which he has pointed out that all the PT-symmetric Hamiltonians regarded so far are actually P-pseudo Hermitian, namely $PHP^{-1} = H^{\dagger}$. Again it is claimed that it is nothing but η-pseudo Hermiticity *i.e.*, $\eta H \eta^{-1} = H^{\dagger}$. By highlighting the concept of pseudo-Hermiticity he has addressed that pseudo-Hermitian is a generalization of Hermiticity.

The purpose of the present article is to construct a transformation which act as parity, imaginary shift and anti-isospectral transformation. Using this transformation, we study the energy spectrum for several solvable potentials.

The organization of the present article is as follows. In Sec. II, we have discussed the anti-isospectral transformation. In Sec. III, we have applied this transformation for several potentials. Sec. IV, is kept for conclusions and discussions.

II. ANTI-ISOSPECTRAL TRANSFORMATION

We propose the transformation $\eta = e^{-i\theta(x-\xi)\frac{d}{dx}}$, $(= e\theta x - \xi p)$, where ξ is scalar θ is real. The above η satisfies the following properties:

$$e^{\theta(x-\xi)p} x \, e^{-\theta(x-\xi)p},$$

$$= x + \theta[(x-\xi)p, x] + \frac{\theta^2}{2!}[(x-\xi)p, [(x-\xi)p, x]] + \ldots \infty$$

$$= (x-\xi) + (-i\theta)(x-\xi) + \frac{(-i\theta)^2}{2!}(x-\xi) + \cdots \infty$$

$$= e^{-i\theta}(x-\xi). \qquad (1)$$

Where we use the relation

$[(x-\xi)p, k(x-\xi)] = -ik(x-\xi)$, k is scalar. By noting that

$$e^{\theta(x-\xi)p} \, x^2 \, e^{-\theta(x-\xi)p}$$

$$= e^{\theta(x-\xi)p} \, x \, e^{-\theta(x-\xi)p} e^{\theta(x-\xi)p} \, x \, e^{-\theta(x-\xi)p}$$

$$= e^{-i\theta}(x-\xi) e^{-i\theta}(x-\xi)$$

$$= e^{-i2\theta}(x-\xi)^2. \qquad (2)$$

Hence by principle of induction,

$$e^{\theta(x-\xi)p}\, x^n\, e^{-\theta(x-\xi)p}$$

$$= e^{-in\theta}(x-\xi)^n.$$

The action of η on the momentum operator p is the following:

$$e^{\theta(x-\xi)p} p\, e^{-\theta(x-\xi)p}$$

$$= x + \theta[(x-\xi)p, p] + \frac{\theta^2}{2!}[(x-\xi)p, [(x-\xi)p, p]] + \ldots\infty$$

$$= p + (i\theta)p + \frac{(i\theta)^2}{2!}p + \cdots\infty$$

$$= e^{i\theta}p. \quad (3)$$

Where we use the relation
$[(x-\xi)p, kp] = ikp$, k is scalar. By noting that

$$e^{\theta(x-\xi)p}\, p^2\, e^{-\theta(x-\xi)p}$$

$$= e^{\theta(x-\xi)p}\, p\, e^{-\theta(x-\xi)p} e^{\theta(x-\xi)p}\, p\, e^{-\theta(x-\xi)p}$$

$$= e^{i\theta}pe^{i\theta}p$$

$$= e^{i2\theta}p^2.$$

Similarly, the action of η on the constant operator c and the potential *V(x)* is the following:

$$e^{\theta(x-\xi)p} c\, e^{-\theta(x-\xi)p} = c, \quad (4)$$

$$e^{\theta(x-\xi)p}V(x)e^{-\theta(x-\xi)p} = V\left(e^{-i\theta}(x-\xi)\right). \quad (5)$$

Using Taylor series expansion, the wave function becomes

$$e^{\theta(x-\xi)p}\psi(x)e^{-\theta(x-\xi)p} = \psi\left(e^{-i\theta}(x-\xi)\right). \quad (6)$$

III. APPLICATION ON POTENTIAL MODEL

We shall now discuss the three choices of η

1. $\eta_1 = e^{-i\pi(x-\xi)\frac{d}{dx}}$ where ξ is real. (7)

2. $\eta_2 = e^{-i2\pi(x-\xi)\frac{d}{dx}}$ where $\xi = i\epsilon$. (8)

3. $\eta_3 = e^{-i\frac{3\pi}{2}(x-\xi)\frac{d}{dx}}$ where $= \epsilon - i\epsilon$. (9)

It is noted that, η_2 is only Hermitian, η_3 can be made Hermitian by multiplying $-i$ ($-i\eta_3$ is Hermitian).

Pöschl-Teller Potential: The generalized Pöschl-Teller Potential [3] is given by

$$V(x) = -V_1 sech_q^2 x - iV_2 sech_q x\, tanh_q x\,,$$

$$(V_1 > 0, q > 0\,). \quad (10)$$

where the deformed type hyperbolic functions are defined as

$$sinh_q x = \frac{e^x - qe^{-x}}{2},\, cosh_q x = \frac{e^x + qe^{-x}}{2},\, tanh_q x = \frac{sinh_q x}{cosh_q x}.$$

The potential given in Eq. (10) is PT-invariant under parity.

$PxP^{-1} = log_e q - x$, also $\eta_1 x\eta_1{}^{-1} = \xi - x$. Now the action on the Hamiltonian $H = -p^2 + V(x)$ is $\eta_1 H\eta_1{}^{-1} = -p^2 + V(\xi - x)$. Hence the Hamiltonian corresponding to the potential Eq. (10) is pseudo Hermitian, $\eta_1 H\eta_1{}^{-1} = H^\dagger$ if $\xi = log_e q$.

Now $\eta_2 x\eta_2{}^{-1} = x + i\epsilon$, which is the imaginary shift operator. Pseudo-Hermitian under η_2, $\eta_2 H\eta_2{}^{-1} = H^\dagger$.

Now we focus on the transformation under η_3, $\eta_3 x\eta_3{}^{-1} = i(x-\epsilon) - \epsilon$.

Potential model	Energy Spectrum	Condition for PT-symmetry
$V_1^T(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)sec^2(x+i\epsilon) + \left(\frac{\alpha^2-\beta^2}{2}\right)\sec(x+i\epsilon)\tan(x+i\epsilon)$	$\left(n + \frac{\alpha+\beta+1}{2}\right)^2$	$\alpha^* = \pm\beta$
$V_1^H(x) = -\left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)sech^2(x+i\epsilon) - i\left(\frac{\alpha^2-\beta^2}{2}\right)\sec\text{h}(x+i\epsilon)\tanh(x+i\epsilon)$	$-\left(n + \frac{\alpha+\beta+1}{2}\right)^2$	$\alpha^* = \pm\alpha$ $\beta^* = \pm\beta$
$V_2^T(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)csc^2(x+i\epsilon) + \left(\frac{\alpha^2-\beta^2}{2}\right)\csc(x+i\epsilon)\cot(x+i\epsilon)$	$\left(n + \frac{\alpha+\beta+1}{2}\right)^2$	$\alpha^* = \pm\alpha$ $\beta^* = \pm\beta$
$V_2^H(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)csch^2(x+i\epsilon) + \left(\frac{\alpha^2-\beta^2}{2}\right)\csc\text{h}(x+i\epsilon)\cot\text{h}(x+i\epsilon)$	$-\left(n + \frac{\alpha+\beta+1}{2}\right)^2$	$\alpha^* = \pm\alpha$ $\beta^* = \pm\beta$
$V_3^T(x) = \left(\beta^2 - \frac{1}{4}\right)sec^2(x+i\epsilon) + \left(\alpha^2 - \frac{1}{4}\right)\csc^2(x+i\epsilon)$	$(2n+\alpha+\beta+1)^2$	$\alpha^* = \pm\alpha$ $\beta^* = \pm\beta$
$V_3^H(x) = -\left(\beta^2 - \frac{1}{4}\right)sech^2(x+i\epsilon) + \left(\alpha^2 - \frac{1}{4}\right)\text{csch}^2(x+i\epsilon)$	$-(2n+\alpha+\beta+1)^2$	$\alpha^* = \pm\alpha$ $\beta^* = \pm\beta$

The solution of the above potentials have been discussed on Ref. [4]. From the above table, we have, $\eta_3 V_k^T(x)\eta_3^{-1} = -V_k^H(x)$, $k = 1, 2, 3$ and $\eta_3 H_k^T(x)\eta_3^{-1} = -H_k^H(x)$, $k = 1,2,3$ where $V_k^T(x)$ is trigonometric type potential and $V_k^H(x)$ is hyperbolic type potential and we use the relation $\cos(iZ) = \cosh Z$, $\sin(iZ) = \mathrm{i}\sinh Z$. Obviously, if E is the value of the potential $V_k^T(x)$, then $-E$ is the value of the potential $-V_k^H(x)$. Again if $\psi(x)$ and $\psi(ix + \epsilon')$ $(\epsilon' = -\varepsilon - i\varepsilon)$ satisfy appropriate boundary condition and $\psi(x)$ is the solution of the Schrödinger equation with potential $V_k^T(x)$ then $\psi(ix + \epsilon')$ is the solution of the Schrödinger equation with potential $-V_k^H(x)$. The new potential $-V_k^H(x)$, generated by the anti-isospectral transformation η_3, is clearly PT-symmetric.

IV. CONCLUSIONS

In this paper, we have discussed three kind of transformations which act as parity, imaginary shift of the coordinate and anti-isospectral. We have applied this transformation on several potential models. We have also discussed the positive and negative energy spectrum under the transformation η_3.

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