Motion with Non-Constant Gravitational Acceleration

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(Received 13 August 2011, accepted 23 September 2011)

Abstract
Standard physics textbooks do not provide a quantitative account of Newtonian two-body gravitational interactions involving explicit non-constant acceleration. This is a gap in contemporary undergraduate textbooks which should be addressed. An accessible treatment is presented in this article.

Keywords: Newtonian gravitation; non-constant acceleration.

Resumen
Los libros de texto estándar de física no proporcionan un análisis cuantitativo explícito sobre interacciones gravitacionales Newtonianas de dos cuerpos sin aceleración constante. Hay una laguna en los libros de texto contemporáneos de pregrado, que deben abordarse. En este artículo se presenta un tratamiento accesible.

Palabras clave: Gravitación Newtoniana, constante de no aceleración.

PACS: 45.20.da, 45.50.Jf **ISSN 1870-9095**

I. INTRODUCTION

The study of gravitational phenomena is essential to gaining a basic understanding of real world physics. The usual approach to Newtonian two-body gravitational interactions in undergraduate studies is to give an account involving constant acceleration (such as a small object falling near the Earth's surface) and does not require explicit use of non-constant acceleration terms (such as a planet orbiting the Sun obeying Kepler's three laws). Despite the fact that circumstances involving non-constant acceleration are far more realistic, standard physics textbooks do not provide treatments involving explicit non-constant acceleration calculations (typical examples include: [1, 2, 3, 4, 5, 6, 7, 8]). Problems with non-constant gravitational acceleration are rarely set in the textbooks (and generally quite old ones) but without providing worked solutions (see, for example: [9, 10, 11]).

The reason for the common absence of these treatments and problems seems to be a view that the mathematics is above a level that is suitable for use in undergraduate courses. Whilst this is the case for first year studies in physics, non-constant gravitational acceleration problems use standard methods for solving ordinary differential equations and is, therefore, suitable for later year studies. The lack of quantitative accounts of Newtonian gravitational interactions with explicit non-constant acceleration constitutes a gap in general undergraduate physics textbooks. This article provides an accessible treatment which, hopefully, will serve to rectify this deficiency in these texts.

II. TWO REPRESENTATIVE SCENARIOS

Detailed solutions to two representative scenarios with explicit non-constant gravitational acceleration are presented below. These solutions are designed to help fill the existing hole in undergraduate textbooks and also show that the level of mathematics is suitable for advanced undergraduate studies.

In both scenarios, the bodies involved are two spheres. The first scenario is where the spheres are of unequal masses. One sphere is fixed and the other accelerates until it makes contact with the fixed sphere and stops. The second scenario is where the spheres are of equal mass. Each sphere accelerates towards the other until they make contact and all motion ceases. The time taken to impact and the kinetic energies involved in both scenarios are calculated.

Some interesting conclusions can be seen from applying these calculations in numerical examples. As an example of the first scenario, if the fixed sphere is chosen to be the Earth and the other sphere to be a small body at an altitude of 1000 kilometers, then the time taken to impact is found (from Eq. (8)) to be about 9 minutes. This time until impact is slightly longer than that which would be found if the gravitational acceleration used was a constant value averaged from the Earth's surface to the chosen altitude.

As an example of the second scenario, suppose each sphere is one metre in radius with a mass of 1000kg, and that their centres are initially 200 meters apart. Then the time taken to meet is found (from Eq. (14)) to be about 100 days. This time interval is quite realistic due to the minuscule magnitude of the mutual gravitational attraction. The time interval is, however, at least an order of magnitude greater than that suggested by some textbooks for this kind of situation [12]. The scenario also provides a good illustration that gravitational potential energy belongs to the system and not individual objects as it is the sum of the final kinetic energies of both spheres that is found to be equal to the magnitude of the change in the gravitational potential energy of the system consisting of both spheres.

III. ONE BODY STATIONARY

Assume two solid spherical bodies of uniform densities (denoted A and B) which have radii R_A, R_B and masses m_A and m_B respectively. The spheres have no angular momentum and gravity is the only force acting. If m_A is many orders of magnitude greater than m_B then we may assume sphere A to be stationary. Let the motion of sphere B be with respect to an external inertial frame of reference S in which sphere A is at rest with its centre fixing the origin of the x-axis of a Cartesian coordinate system, as depicted in Fig. 1.

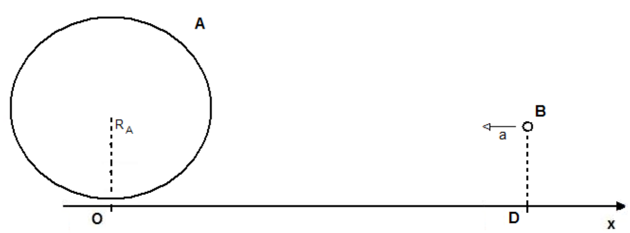

FIGURE 1. Small sphere accelerating towards a massive sphere.

Initially, let the distance between the centres of the spheres be D with sphere B also at rest in frame S. At time t > 0, sphere B will be moving with non-constant acceleration of magnitude a in the negative x-direction (*i.e*. towards sphere A) until such time as the surfaces of the two spheres make contact. The acceleration of sphere B during this interval will be:

$$a = (d^2x/dt^2) = -(Gm_A / x^2) = -(k^2/x^2), \qquad (1)$$

where $k^2 = Gm_A$ and $x > (R_A + R_B)$. Now let v be the velocity of sphere B, then we have:

$$(d^2x/dt^2) = (dv/dt) = (dv/dx)\,(dx/dt) = (dv/dx)\,v. \qquad (2)$$

Combining Eqs. (1) and (2) gives:

$$v\,(dv/dx) = -(k^2/x^2). \qquad (3)$$

Since Eq. (3) is separable, we find:

$$\int v\,dv = -k^2 \int x^{-2}\,dx.$$

Integrating produces the result:
$½v^2 = (k^2/x) + C_1$, where C_1 is a constant of integration.

At t = 0, v = 0 and x = D $\Rightarrow$ $C_1 = -(k^2/D)$. Therefore

$$½v^2 = (k^2/x) - (k^2/D). \qquad (4)$$

Solving for v, it can be seen that:

$$v = (dx/dt) = \pm\ [2k^2\{(1/x) - (1/D)\}]^{½} = \pm\sqrt{2}\,k\,\{(D - x)/xD\}^{½}.$$

Since sphere B is moving in the negative x-direction, choose the minus sign. Then we have:

$$(2/D)^{½}\,k \int dt = -\int [x^{½}/(D - x)^{½}]\,dx. \qquad (5)$$

Using standard integration tables [13] and Eq. (5), we get:

$$(2/D)^{½}\,kt + C_2 = x^{½}(D - x)^{½} - D\arctan[x^{½}/(D - x)^{½}], \qquad (6)$$

where C_2 = constant. When t = 0, x = D $\Rightarrow$ $C_2 = -(½\pi D)$.

Then rewriting Eq. (6) gives:

$$t = (D/2Gm_A)^{½}\{x^{½}(D - x)^{½} - D\arctan[x^{½}/(D - x)^{½}] + (½\pi D)\}. \qquad (7)$$

Eq. (7) cannot be re-arranged to express x only as a function of t. However, it is clear that the path of sphere B will be along an (imagined) straight line joining the centres of the spheres.

The equation for the time taken for the surface of sphere B to impact against the surface of sphere A, *i.e*. at $x = (R_A + R_B)$, may be accurately approximated as follows. If $R_A \gg R_B$, $x \approx R_A$ and $(D - x) \approx (D - R_A)$. Then we have from Eq. (7):

$$t \approx (D/2Gm_A)^{½}\{[R_A(D - R_A)]^{½} - D\arctan[R_A^{½}/(D - R_A)^{½}] + (½\pi D)\}. \qquad (8)$$

Eq. (4) also readily allows for the calculation of the kinetic energy of sphere B (K_B):

$$K_B = ½\,m_B v^2 = (k^2 m_B/x) - (k^2 m_B/D) = Gm_A m_B\{(1/x) - (1/D)\}.$$

When sphere B impacts with sphere A, $x = (R_A + R_B)$ and its final kinetic energy (K_B^f) will be:

$$K_B^{\,f} = Gm_A m_B\{ [1/(R_A + R_B)] - (1/D) \}$$
$$= Gm_A m_B \{[D - (R_A+R_B)]/D\,(R_A+R_B)\}.$$

Since it was assumed that sphere A is at rest, $K_B^{\,f}$ is equal to the magnitude of the change in the gravitational potential energy of the system (consisting of both spheres and their gravitational fields) when sphere B has moved from x = D to $x = (R_A + R_B)$.

IV. BOTH BODIES IN MOTION

Again assume two spheres (A and B) in the inertial frame of reference S, initially at rest, separated by a centre-to-centre distance D, and with the initial position of sphere A's centre fixing the origin. In order to simplify this problem, the spheres are assumed to have equal radii (R) and equal masses (m). The spheres are only acted upon by their mutual gravitational attraction and consequently after time t = 0, neither sphere will be stationary in the inertial frame S. At time t > 0, the spheres will be moving with equal, non-constant acceleration of magnitude a towards each other until their surfaces make contact and motion ceases. The point of contact will be x = (D/2), which is the centre-of-mass point for the system. Although meeting at the centre-of-mass is a feature for bodies with unequal masses as well, it can be seen to directly follow in this case from both spheres being the same mass, starting from rest and having the same value of acceleration at a given time.

Let the distance travelled by each sphere at any particular time be s, as shown in Fig. 2.

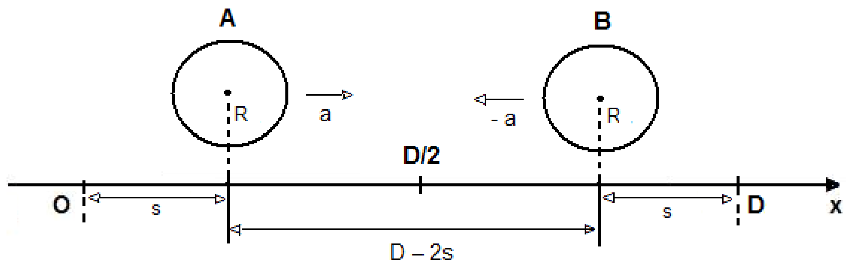

FIGURE 2. Two spheres of equal size and mass accelerating towards each other at time t > 0.

Then at time t, the distance between the centres of the spheres will be (D – 2s). Applying this to sphere A gives:

$$a = (dv/dt) = Gm / (D-2x)^2 = \kappa^2/(D - 2x)^2 \quad , \qquad (9)$$

where $0 \le x < (½D - R)$ is the displacement of A at time t and $\kappa^2 = Gm$. Following the same procedure as used in Section III, we get from Eq. (9):

$$\int v\, dv = \kappa^2 \int dx/(D-2x)^2.$$

Integrating gives:

$$½v^2 = [\kappa^2/2(D-2x)] + C_3,$$

where C_3 = constant. For sphere A, when t = 0, v = 0 and x = 0 $\Rightarrow C_3 = -(\kappa^2/2D)$. Therefore:

$$½v^2 = [\kappa^2/2(D-2x)] - (\kappa^2/2D). \qquad (10)$$

Solving for v, it can be seen that:

$$v = (dx/dt) = \pm \{[\kappa^2/(D-2x)]-(\kappa^2/D)\}^{½}$$
$$= \pm (2/D)^{½}\, \kappa\, \{x/(D-2x)\}^{½}, \qquad (11)$$

Choose the positive square root in Eq. (11) as sphere A is moving in the + x-direction. Then we get:

$$(2/D)^{½}\, \kappa \int dt = \int [(D-2x)^{½}/x^{½}]\, dx.$$

Integrating we find:

$$(2/D)^{½}\, \kappa\, t + C_4 =$$
$$x^{½}(D-2x)^{½} + (D/\sqrt{2}) \arctan [\sqrt{2}\, x^{½}/(D-2x)^{½}], \qquad (12)$$

where C_4 = constant. When t = 0, x = 0 $\Rightarrow C_4 = 0$. Therefore, from Eq. (12):

$$t = (D/2Gm)^{½} \{x^{½} (D-2x)^{½}$$
$$+ (D/\sqrt{2}) \arctan [\sqrt{2}\, x^{½}/(D-2x)^{½}]\}. \qquad (13)$$

Eq. (13) also cannot be re-arranged to express x only as a function of t but it is apparent that both spheres will follow an (imagined) straight line joining their centres.

The final position of sphere A's centre will be at x = (½D–R) so that (D–2x) = 2R. Using Eq. (13), the value of time t when this final position is reached will be:

$$t = (D/2Gm)^{½} \{(RD-2R^2)^{½} +$$
$$(D/\sqrt{2}) \arctan ([(D-2R)/2R]^{½})\}.$$

If D >> R, the time taken for the spheres to meet will be approximately given by:

$$t \approx (D/2Gm)^{½}\{(RD)^{½} + (D/\sqrt{2}) \arctan[(D/2R)^{½}\}. \qquad (14)$$

Eq. (10) allows the kinetic energy of sphere A (K_A) to be easily calculated (which is, of course, also equal to the kinetic energy of sphere B):

$$K_A = \frac{1}{2}\, m\, v^2 = [\kappa^2\, m/2(D-2x)]-(\kappa^2 m/2D)$$
$$= \frac{1}{2}\, Gm^2\, \{[(1/(D-2x)]-(1/D)\}.$$

When the surfaces of the two spheres make contact, x = (½D–R) for sphere A. Then the final kinetic energy of sphere A (K_A^f) will be:

$$K_A^f = \frac{1}{2}\, Gm^2\, \{(1/2R)-(1/D)\} = \frac{1}{4}\, Gm^2\, \{(D-2R)/RD\} = K_B^f$$

where K_B^f is the final kinetic energy of sphere B. Since both spheres move in this case, it is the sum of their final kinetic energies that will be equal to the magnitude of the change in the gravitational potential energy of the system (consisting of both spheres and their gravitational fields) (ΔU), *i.e*.

$$\Delta U = \frac{1}{2}\, Gm^2\, \{(D-2R)/RD\} = K_A^f + K_B^f.$$

This is a good illustration that gravitational potential energy belongs to the system, not the individual objects.

V. CONCLUSIONS

This article provides an accessible treatment which is aimed at rectifying the absence in standard physics textbooks of treatments involving explicit non-constant Newtonian gravitational acceleration. The two scenarios presented show that this is an appropriate topic for later year undergraduate studies. Only accounts with non-constant acceleration give correct times for the motion of bodies acting under their mutual gravitational attraction.

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