Dynamics of a particle sliding down a smooth exponential incline

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Abstract

Problems on motion of particles sliding down inclines of specific geometrical shapes are found in college physics curricula. This paper examines the motion of a particle sliding down a frictionless exponential incline under gravity. Dynamical variables such as velocity, acceleration and jerk vectors are calculated as functions of the vertical coordinate and hodographs of these vectors are constructed. The velocity of the particle commences as a null vector having a definite orientation and increases in magnitude until it attains a maximum limiting value and becomes horizontal. Its hodograph rotates anti-clockwise through an acute angle. The acceleration vector begins as a finite vector and ends as a null vector having a definite orientation after attaining its greatest magnitude somewhere during the early phase of the motion. It also rotates anti-clockwise, at a rate faster than the velocity vector. Finally, the jerk vector begins and ends as null vectors having definite orientations, attaining its greatest magnitude during the initial phase of the motion. It too, rotates counter-clockwise, with the fastest rate among all three vectors. This problem shares resemblances with Galileo's historical inertia experiments.

Keywords: Exponential incline, Dynamical variables, Jerk vector, Hodographs.

Resumen

En los programas de física universitaria se encuentran problemas sobre el movimiento de partículas que se deslizan por pendientes inclinadas de formas geométricas específicas. Este artículo examina el movimiento de una partícula deslizándose por una pendiente exponencial sin fricción debido a la gravedad. Las variables dinámicas tales como la velocidad, aceleración y los vectores jerk se calculan como funciones de las coordenadas verticales y hodógrafas de estos vectores. La velocidad de la partícula comienza como un vector nulo que tiene una orientación definida y aumenta en magnitud hasta que se alcanza un valor máximo que limita y se convierte en horizontal. Su hodógrafa gira hacia la izquierda a través de un ángulo agudo. El vector aceleración comienza como un vector finito y termina como un vector nulo que tiene una orientación definida después de alcanzar su mayor magnitud en algún lugar durante la fase temprana del movimiento. También gira en sentido anti horario, en una tasa más rápida que el vector de velocidad. Finalmente, el vector jerk comienza y termina como vectores nulos teniendo orientaciones definidas, alcanzando su mayor magnitud durante la fase inicial del movimiento. También, gira hacia la izquierda, con la tasa más rápida entre los tres vectores. En este problema se comparte semejanzas con experimentos inercia histórica de Galileo.

Palabras clave: Inclinación exponencial, Variables dinámicas, Vector Jerk, Hodógrafas.

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I. INTRODUCTION

Problems on motion of particles sliding down inclines of specific geometrical shapes are commonly found in college physics curricula. They constitute examples of constrained motion which call for constraint conditions apart from equations of motion provided by Newton's second law. Examples are categorized as follows. (1) *A particle sliding, without friction, down an inclined plane*. This is the simplest problem which leads to a uniformly accelerated motion. (2) *A particle sliding down a frictionless hemispherical dome*. This problem asks one to find the location where the particle leaves the surface. (3) *A particle sliding down a frictionless hemispherical bowl*. This problem is equivalent to a simple pendulum and results in periodic motion. (4) *A particle sliding, without friction, down an inverted cycloid*. This is the famous Brachistochrone problem which engaged the minds of Galileo, Newton, the Bernoulli brothers, Leibniz and L'Hospital [1]. In this paper, we study the dynamics of *a particle sliding down a frictionless exponential incline under gravity*. We examine the various dynamical variables associated with this motion, including the velocity, acceleration and jerk vectors, along with kinetic and potential energies, curvature and centripetal force, as the particle descends under gravity.

Amongst the dynamical variables, we include the *jerk*, which is the derivative of the acceleration vector with respect to time. The jerk vector has recently been studied in projectile motion [2], Brachistochrone motion [1] and motion of a charged particle [3, 4]. We further construct hodographs of the dynamical vectors obtained in this problem. The *hodograph* is the locus of a vector drawn from a fixed point [5]. The hodographs of the velocity, acceleration and jerk vectors have been studies for projectile motion [5] and planetary motion [6].

II. THE PROBLEM

Consider the problem of a particle descending, without friction, down an exponential incline, under gravity. The equation of the incline is given by

$$y = Ae^{-\alpha x}\ , \qquad (1)$$

where A and α are constants. The particle of mass m slides down from the top of the incline $y = A$ at time $t = 0$. The horizontal and vertical components of the velocity are:

$$v_x = \frac{dx}{dt} = \dot{x}, \qquad (2)$$

and

$$v_y = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\alpha Ae^{-\alpha x}\dot{x}. \qquad (3)$$

The kinetic and potential energies of the particle are:

$$T = \frac{1}{2}m\left(v_x^2 + v_y^2\right) = \frac{1}{2}m(1 + \alpha^2A^2e^{-2\alpha x})\dot{x}^2, \qquad (4)$$

and

$$V = mgAe^{-\alpha x}, \qquad (5)$$

where g is the acceleration due to gravity. From the principle of conservation of energy $T + V = T_0 + V_0$, one obtains the equation of motion in *x*:

$$(1 + \alpha^2A^2e^{-2\alpha x})\dot{x}^2 + 2gAe^{-\alpha x} = 2gA. \qquad (6)$$

Eq. (6) is not readily integrable in the closed form. However, the problem simplifies immensely when rendered in terms of the vertical coordinate *y*. The energy equation gives the speed of the particle at any *y*:

$$v = \sqrt{2g(A-y)}. \qquad (7)$$

If θ is the slope angle of the velocity, then

$$tan\theta = \frac{dy}{dx} = \frac{v_y}{v_x} = -\alpha y. \qquad (8)$$

From trigonometry, we have

$$cos\theta = \frac{1}{\sqrt{1+tan^2\theta}} = \frac{1}{\sqrt{1+\alpha^2y^2}}\ , \qquad (9)$$

and

$$sin\theta = \frac{tan\theta}{\sqrt{1+tan^2\theta}} = -\frac{\alpha y}{\sqrt{1+\alpha^2y^2}}\ . \qquad (10)$$

Thus

$$v_x = vcos\theta = \sqrt{\frac{2g(A-y)}{1+\alpha^2y^2}}\ , \qquad (11)$$

and

$$v_y = vsin\theta = -\alpha y\sqrt{\frac{2g(A-y)}{1+\alpha^2y^2}}\ . \qquad (12).$$

Initially, at $y = A$, v_x and v_y are both zero, and $\theta = tan^{-1}(-\alpha A)$. As $y \to 0$, $v_y \to 0$, $\theta \to 0$, and the velocity is entirely horizontal with the limiting value $v_x \to \sqrt{2gA}$.

The *x*- and *y*- components of the acceleration vector can be calculated as follows:

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dy}\frac{dy}{dt} = \frac{dv_x}{dy}v_y, \qquad (13)$$

and

$$a_y = \frac{dv_y}{dt} = \frac{dv_y}{dy}\frac{dy}{dt} = \frac{dv_y}{dy}v_y. \qquad (14)$$

Upon carrying out the differentiations, substituting and simplifying, we arrive at:

$$a_x = g\alpha y\frac{1+2\alpha^2Ay-\alpha^2y^2}{(1+\alpha^2y^2)^2}\ , \qquad (15)$$

and

$$a_y = g\alpha^2y\frac{2A-3y-\alpha^2y^3}{(1+\alpha^2y^2)^2}\ . \qquad (16)$$

If β is the slope of the acceleration vector, then

$$\beta = tan^{-1}\frac{a_y}{a_x} = tan^{-1}\frac{2\alpha A-3\alpha y-\alpha^3y^3}{1+2\alpha^2Ay-\alpha^2y^2}\ . \qquad (17)$$

At $y = A$, $a_x = g\alpha A/(1+\alpha^2A^2)$, $a_y = -g\alpha^2A^2/(1+\alpha^2A^2)$, and $\beta = tan^{-1}(-\alpha A)$. The acceleration vector has the same initial slope as the velocity vector, even though the former is non-zero and the latter is a null-vector. As $y \to 0$, a_x and a_y both tend to zeros and $\beta = tan^{-1}(2\alpha A)$. Thus, even though the acceleration tends to a null-vector, it still retains a well-defined direction.

The *x*- and *y*- components of the jerk vector can be calculated as follows:

$$j_x = \frac{da_x}{dt} = \frac{da_x}{dy}\frac{dy}{dt} = \frac{da_x}{dy}v_y \qquad (18)$$

and

$$j_y = \frac{da_y}{dt} = \frac{da_y}{dy}\frac{dy}{dt} = \frac{da_y}{dy}v_y \qquad (19)$$

Upon carrying out the differentiations, substituting and simplifying, we arrive at:

$$j_x = -g\alpha^2 y\sqrt{2g(A-y)}\frac{1+4\alpha^2Ay-6\alpha^2y^2-4\alpha^4Ay^3+\alpha^4y^4}{(1+\alpha^2y^2)^{7/2}}, \quad (20)$$

and

$$j_y = -2g\alpha^3 y\sqrt{2g(A-y)}\frac{A-3y-3A\alpha^2y^2+\alpha^2y^3}{(1+\alpha^2y^2)^{7/2}}. \quad (21)$$

If γ is the slope of the jerk vector, then

$$\gamma = tan^{-1}\frac{2\alpha\left(A-3y-3A\alpha^2y^2+\alpha^2y^3\right)}{1+4\alpha^2Ay-6\alpha^2y^2-4\alpha^4Ay^3+\alpha^4y^4}. \quad (22)$$

At $y = A$, both j_x and j_y are zeros, but the slope of the jerk vector is $\gamma = tan^{-1}[-4\alpha A/(1-3\alpha^2A^2)]$. As $y \to 0$, j_x and j_y are again both zeros, but $\gamma = tan^{-1}(2\alpha A)$. Thus the jerk is a null-vector at the beginning and end of the motion while maintaining well-defined directions.

The radius of curvature of the path R can be conveniently calculated from the equation

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}. \quad (23)$$

Here, we get

$$R = \frac{\left(1+\alpha^2y^2\right)^{3/2}}{\alpha^2 y}. \quad (24)$$

The reciprocal of R is the curvature κ:

$$\kappa = \frac{1}{R} = \frac{\alpha^2 y}{(1+\alpha^2y^2)^{3/2}}. \quad (25)$$

Eq. (25) can also be obtained from the dynamical variables as [1, 2, 3, 4]:

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}. \quad (26)$$

The centripetal acceleration at any height y is given by:

$$\frac{v^2}{R} = v^2\kappa = \frac{2g\alpha^2(A-y)y}{(1+\alpha^2y^2)^{3/2}}. \quad (27)$$

Clearly, this is zero at both the beginning and end of the motion: $y = A$ and $y = 0$.

III. AN EXAMPLE

We consider an example where $A = 1$ m and $\alpha = 1$ m^{-1}. Then the initial slope of the incline is, by Eq. (8), $\theta = \tan^{-1}(-1) = -45^o$. The Cartesian components of the velocity, acceleration and jerk vectors are, by Eqs. (11), (12), (15), (16), (18) and (19):

$$v_x = \sqrt{\frac{2g(1-y)}{1+y^2}}, \quad (28)$$

$$v_y = -y\sqrt{\frac{2g(1-y)}{1+\alpha^2}}, \quad (29)$$

$$a_x = gy\frac{1+2y-y^2}{(1+y^2)^2}, \quad (30)$$

$$a_y = gy\frac{2-3y-y^3}{(1+y^2)^2}, \quad (31)$$

$$j_x = -gy\sqrt{2g(1-y)}\frac{1+4y-6y^2-4y^3+y^4}{(1+y^2)^{7/2}}, \quad (32)$$

and

$$j_y = -2gy\sqrt{2g(1-y)}\frac{1-3y-3y^2+y^3}{(1+y^2)^{7/2}}, \quad (33)$$

with $g = 9.8m/s^2$. The slope angles of the velocity, acceleration and jerk vectors are, by Eqs. (8), (17) and (22):

$$\theta = tan^{-1}(-y), \quad (34)$$

$$\beta = tan^{-1}\frac{2-3y-y^3}{1+2y-y^2}, \quad (35)$$

and

$$\gamma = tan^{-1}\frac{2\left(1-3y-3y^2+y^3\right)}{1+4y-6y^2-4y^3+y^4}. \quad (36)$$

At time $t = 0$, $v_x = 0$, $v_y = 0$; $a_x = g/2$, $a_y = -g/2$; $j_x = 0$, $j_y = 0$; $\theta = tan^{-1}(-1) = -\pi/4 = -45^o$; $\beta = tan^{-1}(-1) = -\pi/4 = -45^o$; $\gamma = tan^{-1}(-2) = -63.435^o$. As $t \to\infty$, $v_x \to \sqrt{2g} \approx 4.427m/s$, $v_y \to 0$; $a_x \to 0$, $a_y \to 0$; $j_x \to 0$, $j_y \to 0$; $\theta \to tan^{-1}(0) = 0^o$; $\beta \to tan^{-1}(2) \approx 63.435^o$; and $\gamma \to tan^{-1}(2) + \pi \approx 243.435^o$.

Fig. 1 shows the velocity vector and its components at selected altitudes as the particle descends under gravity on the exponential incline. The velocity vector is necessarily tangential to the incline. Also shown in the figure is the hodograph of the velocity vector. The velocity commences from zero and attains its limiting value during which the vector rotates counter-clockwise through an angle of 45^o.

Fig. 2 shows the acceleration vector and its components at selected altitudes as the particle descends under gravity on the exponential incline. The initial magnitude of the acceleration vector is $g/\sqrt{2} \approx 0.707g$. It diminishes to the limiting value of zero as time progresses. Also shown in the figure is the hodograph of the acceleration vector. The vector rotates counter-clockwise through an angle of 45^o + 63.435^o or 108.435^o. Thus it rotates faster than the velocity vector.

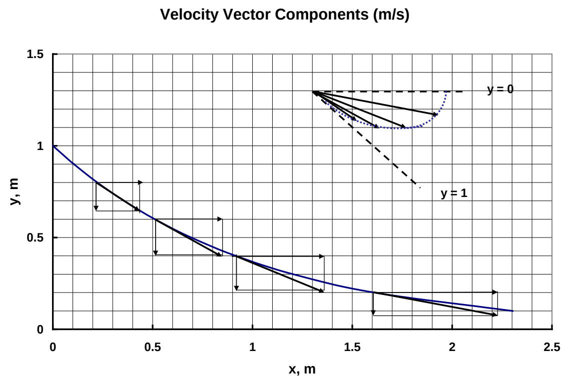

FIGURE 1. Velocity vector of the particle descending down an exponential incline under gravity and its hodograph.

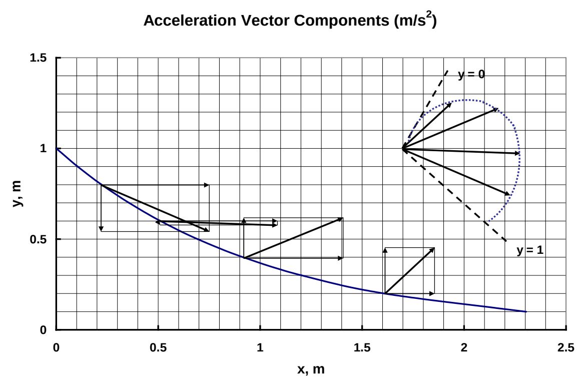

FIGURE 2. Acceleration vector of the particle descending down an exponential incline under gravity and its hodograph.

Fig. 3 shows the jerk vector and its components at selected altitudes as the particle descends under gravity on the exponential incline. The initial and limiting magnitudes of this vector are both zeros. The jerk vector attains its greatest magnitude somewhere during the initial phase of the motion. Also shown in the figure is the hodograph of the jerk vector. The jerk vector rotates counter-clockwise through an angle of 180^o, which is the fastest amongst the three vectors.

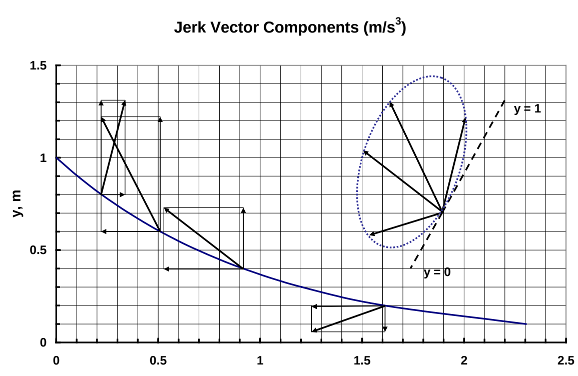

FIGURE 3. Jerk vector of the particle descending down an exponential incline under gravity and its hodograph.

VI. DISCUSSION

Galileo is said to have discovered the law of inertia from his inclined plane experiments, where he rolled marbles down inclined planks placed on a horizontal plane [7, 8]. Planks of various lengths and inclinations were used [7, 8].

Treating the marbles as solid spheres and considering their translational and rotational kinetic energies, the terminal speed of the marble is calculated to be $v = \sqrt{(10/7)gh}$, where h is the height to which one end of the plank is raised. If the marbles can be treated as particles, the terminal speed becomes $v = \sqrt{gh}$, which is exactly the same as terminal speed in the current problem of a particle sliding without friction down the exponential incline. Thus the current problem gives expression to Galileo's inertia experiment if the planks are replaced by exponential inclines and the marbles are treated as particles. Since the terminal speed is proportional to the square root of the initial elevation [7, 8], Galileo arrived at the law of inertia as a limiting case, even though he failed to furnish a categorical statement of the law, which had to wait until Newton's formulation of the laws of motion [9].

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