Simple analytical description of projectile motion in a medium with quadratic drag force

P. S. Chudinov, V. A. Eltyshev, Yu. A. Barykin
Engineering Faculty, Perm State Agricultural Academy, Perm, 614990, Russian Federation.

E-mail: chupet@mail.ru

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Abstract

It is a review of the classic problem of the motion of a point mass (projectile) thrown at an angle to the horizon with allowance for the resistance of the medium. Drag force is accepted as a quadratic function of speed. A full description of the problem is ensured by the simple approximate analytical formulae. This description includes the determining of the basic parameters of the projectile motion. The motions of a baseball is presented as examples.

Keywords: Projectile, quadratic drag force, analytical formulae.

Resumen

Este es un artículo que trata una revisión del problema clásico del movimiento de un punto de masa (proyectil) arrojada en un ángulo al horizonte con una previsión de la resistencia del medio. La fuerza de arrastre es aceptada como una función cuadrática de la velocidad. Una descripción complete del problema es asegurada por las formulas analíticas aproximadas simples. Esta descripción incluye la determinación de los parámetros básicos de movimiento de proyectiles. Los movimientos de una pelota de béisbol se presentan como ejemplos.

Palabras Clave: Proyectil, fuerza de arrastre cuadrática, fórmulas analíticas.

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I. INTRODUCTION

The problem of the motion of a point mass (projectile) thrown at an angle to the horizon has a long history. It is one of the great classical problems. The number of works devoted to this task is immense. This task arouses interest of authors as before [1, 2, 3, 4, 5, 11, 14]. With zero air drag force, the analytic solution is well known. The trajectory of the point mass is a parabola. In situations of practical interest, such as throwing a ball, taking into account the impact of the medium the quadratic resistance law is usually used. In that case the mathematical complexity of the task strongly grows. The problem probably does not have an exact analytic solution. Therefore the attempts are being continued to construct approximate analytical solutions for this problem [6, 7, 8, 9, 10]. For this purpose, various methods are used – both traditional approaches [3, 4, 6, 13], and the modern methods [5]. Besides the description of the projectile motion with a simple approximate analytical formulae under the quadratic air resistance is of great methodological interest.

In [12, 13] comparatively simple approximate analytical formulae have been obtained to study the motion of the projectile in a medium with a quadratic drag force. The proposed analytical solution differs from other solutions by simplicity of formulae, ease of use and high accuracy. All required parameters are determined directly from the initial conditions of projectile motion - the initial velocity and angle of throwing. The proposed formulae make it possible to study the motion of a point mass in a medium with the resistance in the way it is done for the case without drag.

In this article the term “point mass” means the center of mass of a smooth spherical object of finite radius r and cross-sectional area $S = \pi r^2$. The conditions of applicability of the quadratic resistance law are deemed to be fulfilled, *i.e.* Reynolds number Re lies within $1\times10^3 < \text{Re} < 2\times10^5$ [4]. These values correspond to the velocity of motion of a projectile, lying in the range between 0.25 m/s and 53 m/s.

The aim of the present work is to give a simple formula for the construction of the trajectory of the projectile motion with quadratic air resistance, available to senior pupils and first-year undergraduates.

II. EQUATIONS OF MOTION

Suppose that the force of gravity affects the point mass together with the force of air resistance $\boldsymbol{R}$ (Fig. 1), which is proportional to the square of the velocity of the point and is directed opposite the velocity vector. For the convenience of further calculations, the drag force will be written as $R = mgkV^2$. Here m is the mass of the projectile, g is the acceleration due to gravity, k is the proportionality factor. Vector equation of the motion of the point mass has the form

$$m\boldsymbol{w} = m\boldsymbol{g} + \boldsymbol{R},$$

where $\boldsymbol{w}$ – acceleration vector of the point mass. Differential equations of the motion, commonly used in ballistics, are as follows [15]

$$\frac{dV}{dt} = -g\sin\theta - gkV^2,\ \frac{d\theta}{dt} = -\frac{g\cos\theta}{V},$$

$$\frac{dx}{dt} = V\cos\theta, \quad \frac{dy}{dt} = V\sin\theta. \qquad (1)$$

Here V is the velocity of the point mass, θ is the angle between the tangent to the trajectory of the point mass and the horizontal, x, y are the Cartesian coordinates of the point mass, and

$$k = \frac{\rho_a c_d S}{2mg} = \frac{1}{V_{term}^2} = const.$$

is the proportionality factor, ρ_a is the air density, c_d is the drag factor for a sphere, and S is the cross-section area of the object (Fig. 1). The first two equations of the system (1) represent the projections of the vector equation of motion for the tangent and principal normal to the trajectory, the other two are kinematic relations connecting the projections of the velocity vector point mass on the axis x, y with derivatives of the coordinates.

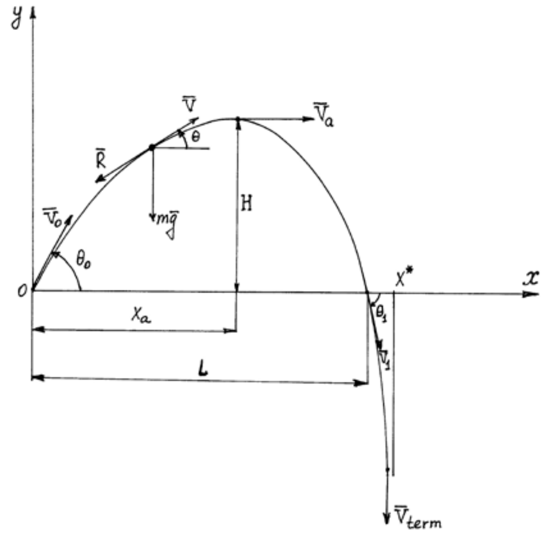

FIGURE 1. Basic motion parameters.

The well-known solution of Eqs. (1) consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures

$$V(\theta) = \frac{V_0\cos\theta_0}{\cos\theta\sqrt{1+kV_0^2\cos^2\theta_0(f(\theta_0)-f(\theta))}},$$

$$f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right). \qquad (2)$$

$$t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} \frac{V}{\cos\theta} d\theta,$$

$$x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta,$$

$$y = y_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 \tan\theta d\theta. \qquad (3)$$

Here V_0 and θ_0 are the initial values of the velocity and the slope of the trajectory respectively, t_0 is the initial value of the time, x_0, y_0 are the initial values of the coordinates of the point mass (usually accepted $t_0 = x_0 = y_0 = 0$). The derivation of the formulae (2) is shown in the well-known monograph [16]. The integrals on the right-hand sides of (3) cannot be expressed in terms of elementary functions. Hence, to determine the variables t, x and y we must either integrate (1) numerically or evaluate the definite integrals (3). Formulae (2) of this solution will be used later.

III. ANALYTICAL FORMULAE FOR DETERMINING THE MAIN PARAMETERS OF MOTION OF THE POINT MASS

Comparatively simple approximate analytical formulae for the main parameters of motion of the projectile are derived in [12, 13]. The four parameters correspond to the top of the trajectory, four – to the point of drop. We will give a complete summary of the formulae for the maximum height of ascent of the point mass H, motion time T, the velocity at the trajectory apex V_a, flight range L, the time of ascent t_a, the abscissa of the trajectory apex x_a, impact angle with respect to the horizontal θ_1 and the final velocity V_1. These formulae are summarized in the right column of Table I. In the left column of this Table I are presented similar formulae of the parabolic theory for comparison.

TABLE I. Analytical formulae for the main parameters.

No drag (R = 0)	**Quadratic drag force ($R = mgkV^2$)**
$H = \frac{V_0^2\sin^2\theta_0}{2g}$	$H = \frac{V_0^2\sin^2\theta_0}{g(2 + kV_0^2\sin\theta_0)}$
$T = 2\frac{V_0\sin\theta_0}{g} = 2\sqrt{\frac{2H}{g}}$	$T = 2\sqrt{\frac{2H}{g}}$
$V_a = V_0\cos\theta_0$	$V_a = \frac{V_0\cos\theta_0}{\sqrt{1 + kV_0^2(\sin\theta_0 + \cos^2\theta_0\ln\tan(\frac{\theta_0}{2} + \frac{\pi}{4}))}}$

$L=\frac{1}{g}V_0^2\sin2\theta_0=V_aT$	$L=V_aT$
$t_a=\frac{V_0\sin\theta_0}{g}=\frac{T}{2}$	$t_a=\frac{T-kHV_a}{2}$
$x_a=\frac{L}{2}=\sqrt{LH\cot\theta_0}$	$x_a=\sqrt{LH\cot\theta_0}$
$\theta_1=-\theta_0=-\arctan\left[\frac{LH}{(L-x_a)^2}\right]$	$\theta_1=-\arctan\left[\frac{LH}{(L-x_a)^2}\right]$
$V_1=V_0$	$V_1=V(\theta_1)$

↑

These formulae enable us to calculate the basic parameters of motion of a point mass directly from the initial data V_0, θ_0, as in the theory of parabolic motion. With zero drag ($k = 0$), the proposed formulae go over into the respective formulae of point mass parabolic motion theory. We note that the structure of the formulae for the parameters is the same at the movement with resistance and at the movement without resistance.

As an example of the use of present formulae we calculated the motion of a baseball with the following initial conditions

$$V_0=40\text{ m/s},\ \theta_0=45°,\ k=0.000625\text{ s}^2/\text{m}^2,\ g=9.81\text{ m/s}^2. \quad (4)$$

TABLE II. Comparison of numerical and analytical calculations.

Parameter	**Numerical value**	**Analytical value**	**Error (%)**
H, m	29.81	30.12	+1.0
T, s	4.91	4.96	+1.0
V_a, m/s	19.30	19.30	0
L, m	96.07	95.68	-0.4
t_a, s	2.31	2.30	-0.4
x_a, m	53.02	53.68	+1.2
θ_1, deg	-57.27	-58.55	+2.2
V_1, m/s	25.53	26.00	+1.8

The results of calculations are recorded in Table II. The second column shows the values of parameters obtained by numerical integration of the motion equations (1) by the fourth-order Runge-Kutta method. The third column contains the values calculated by present formulae from the Table I. The deviations from the exact values of parameters are shown in the fourth column of the Table 2. Tabulated data show that the values of basic parameters of the projectile motion (flight range L, motion time T, height H) calculated by analytical formulae differ from the exact values no more than 1%.

Fig. 2 is an interesting geometric picture for Table II. If we use motion parameters L, H, x_a to construct the ABC triangle with the height BD = LH, segments AD = x_a^2 and CD = $(L-x_a)^2$, then in this triangle ∠A ≈θ_0, ∠C ≈θ_1. Thus, for the values L = 96.07, H = 29.81, x_a = 53.02 we have: ∠A = 45.5°, ∠C = 57.1°. Recall that the exact values of angles are ∠A=45°, ∠C = 57.3°.

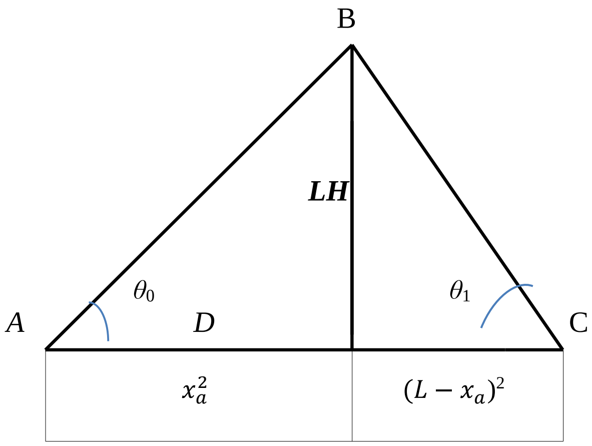

FIGURE 2. Motion parameters.

Present formulae make it possible to obtain simple approximate analytical expressions for the basic functional relationships of the problem $y(x)$, $y(t)$, $y(\theta)$, $x(t)$, $x(\theta)$, $t(\theta)$ [12, 13].

We construct the first of these dependencies. In the absence of a drag force, the trajectory equation of a point mass is a parabola

$$y(x)=x\tan\theta_0-\frac{gx^2}{2V_0^2\cos^2\theta_0}.$$

Using parameters H, L, x_a from the left column of Table I, this equation can be written as

$$y(x)=\frac{Hx(L-x)}{x_a^2}. \quad (5)$$

When the point mass is under a drag force, the trajectory becomes asymmetrical. The top of the trajectory is shifted toward the point of incidence. In addition, a vertical asymptote appears near the trajectory. Taking these circumstances into account, we shall construct the function $y(x)$ as [12]

$$y(x) = \frac{Hx(L-x)}{x_a^2+(L-2x_a)x}. \qquad (6)$$

The constructed dependence *y(x)* provides the shift of the apex of the trajectory to the right and has a vertical asymptote. In the case of no drag $L = 2x_a$, relationship (6) goes over into (5).

We note the remarkable property of the formula (6). We substitute the exact values of the parameters L, H, x_a, obtained by numerical integration of the system (1), into the formula (6). Then the numerical trajectory and the analytical trajectory constructed by means of the formula (6) are identically the same. This means that formula (6) approximate absolutely precisely projectile's trajectory which are numerically constructed with using equations (1) at any values of the initial conditions V_0, θ_0.

Based upon Eqs. from Table I and (6) an approximate trajectory was constructed. It is shown in Fig. 3 (dashed line). The same values (4) were used for the calculations. Thick solid line in this figure is obtained by numerical integration of motion equations (1) with the aid of the 4-*th* order Runge-Kutta method. As it can be seen from the figure, the analytical solution (6) and a numerical solution are almost the same. Dotted line in this figure is constructed in the absence of air resistance.

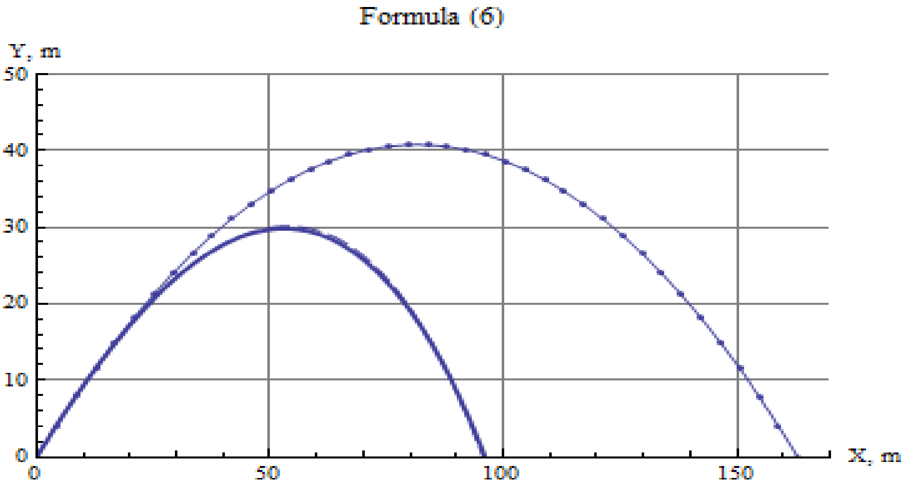

FIGURE 3. The graph of the trajectory *y= y(x).*

Thus, simple formulae containing elementary functions are used to construct a projectile trajectory with quadratic law of air resistance. It can be implemented even on a standard calculator. Lately many authors [9, 10, 11] have used the Lambert *W* function to study the projectile motion with resistance. But this relatively "new" function is not available on a calculator.

For a baseball the typical values of the drag force coefficient k is about 0.0005 ÷ 0.0006 s^2/m^2 [6, 8]. We introduce the notation $p = kV_0^2$. The dimensionless parameter p has the following physical meaning - it is the ratio of air resistance to the weight of the projectile at the beginning of the movement. Formulae from the Table I have a bounded region of application. The main characteristics of the motion H, T, V_a, L, t_a, x_a, θ_1, V_1 have accuracy to within 2 - 3% for values of the launch angle, for initial velocity and for the parameter p from ranges

$$0° \le \theta_0 \le 70°,\ 0 \le V_0 \le 50 \text{ m/s},\ 0 \le p \le 1.5. \qquad (7)$$

The some transformation of the proposed formulae [14] makes it possible to improve the accuracy of calculating the main parameters. Now it is possible to construct the trajectory in the entire range of launch angles and at values of the initial velocity and the parameter p

$$0° \le \theta_0 \le 90°,\ 0 \le V_0 \le 80 \text{ m/s},\ 0 \le p \le 4.$$

IV. CONCLUSION

The proposed approach is based on the theory of the parabolic motion of the projectile. The use of analytical formulae make it possible to simplify significantly a qualitative analysis of the projectile motion with quadratic drag force. All basic parameters are described by simple analytic formulae. Moreover, numerical values of the sought variables are determined with acceptable accuracy. Thus, proposed formulae make it possible to study projectile motion with quadratic drag force even for senior pupils.

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