Solvable potentials with supersymmetric partners and potential algebra

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Abstract

An algebraic method of three constructing potentials for which the Schrödinger equation can be solved exactly is presented. A form of the generators of SL(2,C) potential algebra has been employed to the problem. These potentials have been expressed as supersymmetric partner potentials. Finally, the results are compared with ones obtain before.

Keywords: Potential Algebra, Superpartner, Factorization energy.

Resumen

Se presenta un método algebraico de tres potenciales de la construcción para los que la ecuación de Schrödinger se puede resolver exactamente. Una forma de los generadores del álgebra SL(2,C) del espacio potencial se ha empleado para el problema. Estos potenciales se han expresado como potenciales socios con supersimetría. Por último, los resultados se comparan con los obtenidos anteriormente.

Palabras clave: Álgebra potencial, Supercompañero, energía de factorización.

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I. INTRODUCTION

The Hermitian quantum mechanics is a well-developed framework because a Hermitian Hamiltonian leads to a real spectrum. However, a decade ago, it was observed that a large class of non-Hermitian Hamiltonians possess real spectra. Imaginary potential usually appears in a system to describe physical processes phenomenologically due to its simplicity, which has been investigated under the non-Hermitian quantum mechanics framework. The study of exactly solvable potentials has generated a lot of interest since the early development of supersymmetric quantum mechanics [1, 2, 3, 4, 5]. Originally, the idea of supersymmetry first appeared in field theories in terms of bosonic and fermionic fields. The eigenvalues and eigenvectors of these potentials can be derived by using the methods of supersymmetric quantum mechanics. The majority of these potentials have also been shown to possess a potential algebra and hence are also solvable by group theoretical techniques. Recently, Lie algebraic techniques [6, 7, 8, 9, 10, 11, 12, 13] have been used extensively to obtain the spectra of various physical systems such as rotation vibration spectra in molecules [14] and collective states in nuclei [15]. The introduction of the deformation parameter may serve as an additional parameter in describing inter-atomic interactions.

In this paper, we discuss three exactly solvable complex potentials by using a potential algebraic approach based on complex Lie algebra SL(2,C). We shall show that these potentials are generated by the complex superpotential and the energy spectrum of these potentials become real for particular choice of the parameter. We shall also show that there is an intimate relationship between potential algebra and supersymmetric quantum mechanics.

The present study is organized as follows. To make this work self contained, we first give a brief review of the SL(2,C) potential algebra in Sec. II. Sec. III is devoted to express the Scarf potential, the Pöschl-Teller potential and the Morse potentialin terms of supersymmetric partners. Finally, we make a few concluding remarks in Sec. IV.

II. SL(2,C) POTENTIAL ALGEBRA

The SL(2,C) potential algebra is described by the three generators J_0, $J_\pm$ [11, 12, 13]. These generators are connected by:

$$[J_{0,}, J_\pm] = \pm J_{\pm,} \quad [J_+, J_-] = -2J_0. \tag{1}$$

The Casimir operator of this structure is given by

$$J^2 = -J_\pm J_\mp + J_0^2 \mp J_0 \tag{2}$$

The eigenstate of J_0 and J^2 can be denoted by

$$J_0\left|jk\right\rangle = k\left|jk\right\rangle,\quad k = j, j+1, \ldots\ldots\ldots\ldots\ldots \tag{3}$$

$$J^2\left|jk\right\rangle = j(j-1)\left|jk\right\rangle,\quad k = j, j+1, \ldots\ldots\ldots\ldots\ldots \tag{4}$$

and $j = j_1 + ij_2$, $k = k_1 + ik_2$, $k_1 = j_1 + n$, $k_2 = j_2$ where j_1, j_2, k_1, k_2 are real numbers and n is natural numbers. The states with $j = k$ (*i. e.* $n = 0$) satisfy the equation $J_-\left|jj\right\rangle = 0$, while those with higher values of n can be obtained from them by repeated applications of j_+ and use of the relation $J_+\left|jk\right\rangle \propto j\left|jk+1\right\rangle$.

The differential realization of the above generators are

$$J_0 = -i\frac{\partial}{\partial\phi},$$

$$J_\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} + \left(i\frac{\partial}{\partial\phi} \mp \frac{1}{2}\right)f(x) + g(x)\right], \tag{5}$$

where $0 \le \phi \le 2\pi$, x is real and $f(x), g(x)$ are complex valued functions satisfy

$$\frac{df}{dx} = 1 - f^2,\quad \frac{dg}{dx} = -fg\,. \tag{6}$$

Using Eqns. (2) and (5) one can obtain

$$J^2 = \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2}{\partial\phi^2} + \frac{1}{4}\right)\frac{df}{dx} + 2i\frac{\partial}{\partial\phi}\frac{dg}{dx} - g^2 - \frac{1}{4}\,. \tag{7}$$

The differential realization (7) can be used to derive the second order differential equations. Those differential equations can be expressed in terms of Casimir operator

$$H = J^2,\ H\left|jk\right\rangle = j(j-1)\left|jk\right\rangle. \tag{8}$$

Let us consider the basis function,

$$\left|jk\right\rangle = \Psi_{jk}(x,\phi) = \psi_{jk}\frac{e^{ik\phi}}{\sqrt{2\pi}}\,. \tag{9}$$

It follows that the functions $\psi_{jk}(x)$ satisfies the Schrödinger equation

$$\psi''_{jk} + V_k\psi_{jk} = -\left(j - \frac{1}{2}\right)^2\psi_{jk}$$

$$-\psi''^{(k)}{}_n + V_k\psi^{(k)}{}_n = -E_n^k\psi_n, \tag{10}$$

where $\psi_{jk}(x) = \psi_n^{(k)}(x)$. The family of potentials $V_k(x)$, is represented by

$$V_k(x) = \left(\frac{1}{4} - k^2\right)\frac{df}{dx} + 2k\frac{dg}{dx} + g^2, \tag{11}$$

and the energy eigenvalues are given by

$$E_n^{(k)} = -\left(k_1 + ik_2 - n - \frac{1}{2}\right)^2. \tag{12}$$

Solving the differential equation $J_-\psi_0^{(k)}(x) = 0$, the eigenfunctions $\psi_0^{(k)}(x)$ are easily obtained. Remaining eigenfunctions are obtained by successive application of J_+ on $\psi_0^{(k)}(x)$. For bound states $(\psi_n^{(k)}(\pm\infty) \to 0)$, n is restricted to the range $n = 0,1,2,\ldots..n_{\max} < k_1 - \frac{1}{2}$.

The solutions of the equation (6) are

$$f(x) = \tanh_q(x' - i\sigma),\ \text{and}$$

$$g(x) = (\delta_1 + i\delta_2)\sec h_q(x' - i\sigma), \tag{13}$$

$$f(x) = \coth_q(x' - i\sigma),$$

$$g(x) = (\delta_1 + i\delta_2)\cos ech_q(x' - i\sigma), \tag{14}$$

$$f(x) = \lambda\ \text{and}\ g(x) = (\delta_1 + i\delta_2)e^{-\lambda x}, \tag{15}$$

where $q > 0, c, \delta_1, \delta_2 (\neq 0)$ are real, $\lambda = \pm 1$, $-\frac{\pi}{4} \le \sigma \le \frac{\pi}{4}$, $x' = x - c$ and the deformed hyperbolic functions are defined as: $\sinh_q x = \frac{e^x - qe^{-x}}{2}$, $\cosh_q x = \frac{e^x + qe^{-x}}{2}$ and $\tanh_q x = \frac{e^x - qe^{-x}}{e^x + qe^{-x}}$.

Using Eqns. (11), (13), (14) and (15) we have,

$$V_k(x) = \left\{(\delta_1 + i\delta_2)^2 + \left(\frac{1}{4} - (k_1 + ik_2)^2 q\right)\right\} \times$$
$$\sec h_q^2(x' - i\sigma) - 2(k_1 + ik_2)(\delta_1 + i\delta_2) \times$$
$$\sec h_q(x' - i\sigma)\tanh_q(x' - i\sigma), \tag{16}$$

$$V_k(x) = \left\{(\delta_1 + i\delta_2)^2 - \left(\frac{1}{4} - (k_1 + ik_2)^2 q\right)\right\} \times$$
$$\cos ech_q^2(x' - i\sigma) - 2(k_1 + ik_2)(\delta_1 + i\delta_2) \times$$
$$\cos ech_q(x' - i\sigma)\coth_q(x' - i\sigma), \tag{17}$$

$$V_k(x) = (\delta_1 + i\delta_2)^2 e^{-2(x' - i\sigma)} -$$
$$2(k_1 + ik_2)(\delta_1 + i\delta_2)e^{-(x' - i\sigma)}. \tag{18}$$

III. SUPERSYMMETRIC PARTNERS

By suitable choice of superpotential and factorization energy we shall show that $V_k(x)$, can be considered as a special case of the supersymmetric partner potentials $V^{\pm}(x) = W^2(x) \mp W'(x) + E$.

A. Scarf potential

We take the superpotential and factorization energy as:

$$W(x) = \left(k_1 - \frac{1}{2} + ik_2\right)\tanh_q(x' - i\sigma) - (\delta_1 + i\delta_2)\sec h_q(x' - i\sigma)$$
$$= \frac{1}{\cosh_q^2 x - q\sin^2 y}\Big[\cosh_q x' \left\{\left(k_1 - \frac{1}{2}\right)\sinh_q x' - \delta_1 \cos\sigma\right\} + \sin\sigma\{k_2 q\cos\sigma + \delta_2 \sinh_q x'\}\Big] + \frac{i}{\cosh_q^2 x - q\sin^2 y}\Big[\cosh_q x' \{k_2 \sinh_q x' - \delta_2 \cos\sigma\} - \sin\sigma\left\{q\left(k_1 - \frac{1}{2}\right)\cos\sigma + \delta_1 \sinh_q x'\right\}\Big] \tag{19}$$

and

$$E = \left(k_1 - \frac{1}{2}\right)^2 - k_2^2 + i(2k_1k_2 - k_2) \tag{20}$$

In order to obtain real energy spectrum, the parameters must choosen as $\mathrm{Im}(k) = 0$. The superpotential depends on the deformation parameter q and it is absent in the energy equation (20). Using Eqns. (19) and (20) $V^+(x) = W^2(x) - W'(x) + E$

$$= \left\{(\delta_1 + i\delta_2)^2 + \left(\frac{1}{4} - (k_1 + ik_2)^2 q\right)\right\}\times \sec h_q^2(x' - i\sigma) - 2(k_1 + ik_2)(\delta_1 + i\delta_2)\times \sec h_q(x' - i\sigma)\tanh_q(x' - i\sigma). \tag{21}$$

Eq. (21) is the same as Eq. (16) i.e. $V_k(x) = V^+(x)$. Again

$$V^-(x) = W^2(x) + W'(x) + E = \left\{(\delta_1 + i\delta_2)^2 + \left(\frac{1}{4} - (k_1 - 1 + ik_2)^2 q\right)\right\}\times \sec h_q^2(x' - i\sigma) - 2(k_1 - 1 + ik_2)(\delta_1 + i\delta_2)\times \sec h_q(x' - i\sigma)\tanh_q(x' - i\sigma). \tag{22}$$

Hence $V_{(k_1-1,k_2)}(x) = V^-(x)$.

B. Pöschl-Teller potential

We take the superpotential and factorization energy as:

$$W(x) = \left(k_1 - \frac{1}{2} + ik_2\right)\coth_q(x' - i\sigma) - (\delta_1 + i\delta_2)\cos ech_q(x' - i\sigma)$$
$$= \frac{1}{\sinh_q^2 x + q\sin^2 y}\Big[\sinh_q x' \left\{\left(k_1 - \frac{1}{2}\right)\cosh_q x' - \delta_1 \cos\sigma\right\} - \sin\sigma\{k_2 q\cos\sigma - \delta_2 \cosh_q x'\}\Big] + \frac{i}{\sinh_q^2 x + q\sin^2 y}\Big[\sinh_q x' \{k_2 \cosh_q x' - \delta_2 \cos\sigma\} + \sin\sigma\left\{q\left(k_1 - \frac{1}{2}\right)\cos\sigma - \delta_1 \cosh_q x'\right\}\Big], \tag{23}$$

and

$$E = \left(k_1 - \frac{1}{2}\right)^2 - k_2^2 + i(2k_1k_2 - k_2). \tag{24}$$

To obtain real energy, one has to set $\mathrm{Im}(k) = 0$ in (24). Energy spectrum is independent of deformation parameter q. Using Eqns. (23) and (24)

$$V^+(x) = W^2(x) - W'(x) + E = \left\{(\delta_1 + i\delta_2)^2 + \left(\frac{1}{4} - (k_1 + ik_2)^2 q\right)\right\}\times \cos ech_q^2(x' - i\sigma) - 2(k_1 + ik_2)(\delta_1 + i\delta_2)\times \cos ech_q(x' - i\sigma)\coth_q(x' - i\sigma). \tag{25}$$

Eq. (21) is the same as Eq. (16) *i.e.* $V_k(x) = V^+(x)$. Again

$$V^-(x) = W^2(x) + W'(x) + E = \left\{(\delta_1 + i\delta_2)^2 + \left(\frac{1}{4} - (k_1 - 1 + ik_2)^2 q\right)\right\}\times \cos ech_q^2(x' - i\sigma) - 2(k_1 - 1 + ik_2)(\delta_1 + i\delta_2)\times \cos ech_q(x' - i\sigma)\coth_q(x' - i\sigma). \tag{26}$$

Hence $V_{(k_1-1,k_2)}(x) = V^-(x)$.

C. Morse potential

We take the superpotential and factorization energy as:

$$W(x) = \left(k_1 - \frac{1}{2} + ik_2\right) - (\delta_1 + i\delta_2)e^{-(x' - i\sigma)}$$

$$= \left[\left(k_1 - \frac{1}{2} \right) - (\delta_1 \cos\sigma + \delta_2 \sin\sigma) e^{-x'} \right] + i\left[k_2 - (\delta_2 \cos\sigma + \delta_1 \sin\sigma) e^{-x'} \right], \qquad (27)$$

and

$$E = \left(k_1 - \frac{1}{2} \right)^2 - k_2^2 + i(2k_1k_2 - k_2). \qquad (28)$$

To obtain real energy, one has to set $\mathrm{Im}(k) = 0$ in (28). Using Eqns. (27) and (28)

$$\begin{aligned} V^+(x) &= W^2(x) - W'(x) + E \\ &= (\delta_1 + i\delta_2)^2 e^{-2(x'-i\sigma)} - \\ &\quad 2(k_1 + ik_2)(\delta_1 + i\delta_2) e^{-(x'-i\sigma)}. \end{aligned} \qquad (29)$$

Eq. (29) is the same as Eq. (18) *i.e.* $V_k(x) = V^+(x).$ Again

$$\begin{aligned} V^-(x) &= W^2(x) + W'(x) + E \\ &= (\delta_1 + i\delta_2)^2 e^{-2(x'-i\sigma)} \\ &\quad - 2(k_1 - 1 + ik_2)(\delta_1 + i\delta_2) e^{-(x'-i\sigma)}. \end{aligned} \qquad (30)$$

Hence $V_{(k_1-1,k_2)}(x) = V^-(x).$

IV. CONCLUSIONS

In this paper, a study of SL(2,C) potentials through supersymmetric quantum mechanics have been discussed. Three sets SL(2,C) potentials have been expressed in terms complex superpotential and factorization energy. It is shown that, all the potential cases, H_- has one level less than H_+. Also, the results are consistent with ref [16] for $q = 1$, $k_2 = \delta = 0$.

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