

# Pre-calculus determination of the maximum impedance magnitude and its corresponding frequency for a parallel resonant circuit with a low loss inductor and a low loss capacitor



ISSN 1870-9095

**Kenneth V. Cartwright**

*Department of Technology, School of Mathematics, Physics and Technology,  
University of The Bahamas, P.O. Box N4912, Nassau, Bahamas.*

**E-mail:** kenneth.cartwright@ub.edu.bs

(Received 30 May 2022, accepted August 2022)

## Abstract

The parallel resonant circuit with a low loss inductor and a low loss capacitor is analyzed in this paper. The maximum impedance magnitude and the frequency at which it occurs are determined without using calculus. Additionally, the bandwidth is determined. The frequency at which the impedance has zero angle is also found to be the frequency at which the impedance magnitude is maximum. PSpice simulations are performed which verify the theoretical findings.

**Keywords:** Low loss inductor and low loss capacitor, parallel RL-RC resonant circuit, maximum without calculus, PSpice simulation of parallel resonant circuit.

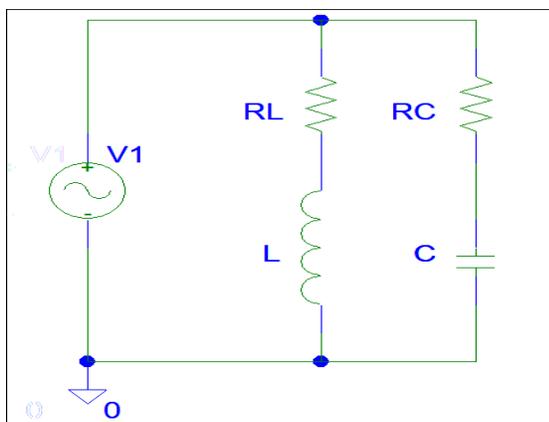
## Resumen

En este trabajo se analiza el circuito resonante en paralelo con un inductor de bajas pérdidas y un condensador de bajas pérdidas. La magnitud de la impedancia máxima y la frecuencia a la que se produce se determinan sin utilizar cálculo. Además, se determina el ancho de banda. También se encuentra que la frecuencia a la que la impedancia tiene un ángulo cero es la frecuencia a la que la magnitud de la impedancia es máxima. Se realizan simulaciones de PSpice que verifican los hallazgos teóricos.

**Palabras clave:** Inductor de bajas pérdidas y condensador de bajas pérdidas, circuito resonante paralelo RL-RC, máximo sin cálculo, simulación PSpice de circuito resonante paralelo.

## I. INTRODUCTION

The complete analysis of the general parallel RL-RC circuit shown in Figure 1 is difficult to find in the literature, except for the trivial case of  $RL=RC=0$ .



**FIGURE 1.** Parallel resonant circuit with a low loss inductor, i.e.,  $RL$  is small, and a low loss capacitor, i.e.,  $RC$  is small. What is meant by small is defined in the text.

In 2015, this was noted by Simon [1], but still remains true today. However, there is one notable exception: the case

where  $RC=0$  and  $RL$  arbitrary was analyzed in [1] and earlier in [2] without using calculus. The purpose of this paper is to provide a pre-calculus analysis for the circuit of Figure 1 when  $RL$  and  $RC$  are small, i.e., the low loss inductor and low loss capacitor case. Specifically, the frequency at which the impedance magnitude is maximum and its maximum value are found without using calculus.

Additionally, the bandwidth will also be determined. Furthermore, the zero phase frequency will be shown to be approximately equal to the frequency at which maximum frequency occurs. Finally, computer simulations will be done to verify the theoretical findings.

## II. ANALYSIS OF THE PARALLEL RESONANT CIRCUIT WHEN $RL$ AND $RC$ ARE SMALL

The total impedance of the parallel resonant circuit of Figure 1 is given by

$$Z_p = \frac{(R_L + i\omega L)(R_C + \frac{1}{i\omega C})}{R_L + i\omega L + R_C + \frac{1}{i\omega C}} \quad (1)$$

where  $i = \sqrt{-1}$  and  $\omega$ (rad/s) is the angular frequency of the applied sinusoid.

For frequencies of interest, the low loss inductor has  $\omega L \gg R_L$ . Similarly, for the low loss capacitor,  $1/(\omega C) \gg R_C$ . Hence,  $R_L + i\omega L \approx i\omega L$  and  $R_C + 1/(i\omega C) \approx 1/(i\omega C)$ . Substituting these approximations into the numerator of Eq. (1) gives

$$Z_p \approx \frac{\frac{L}{C}}{R_L + R_C + i\left(\omega L - \frac{1}{\omega C}\right)}. \quad (2)$$

Furthermore, Eq. (2) can be rewritten as

$$\begin{aligned} Z_p &\approx \frac{(R_L + R_C) \frac{L}{(R_L + R_C)^2 C}}{1 + i\left(\omega \frac{L}{(R_L + R_C)} - \frac{1}{\omega C (R_L + R_C)}\right)} = \\ &\frac{(R_L + R_C) \frac{L}{(R_L + R_C)^2 C}}{1 + i\left(\omega \sqrt{LC} \frac{\sqrt{L}}{\sqrt{C}(R_L + R_C)} - \frac{\sqrt{L}}{\omega \sqrt{LC} \sqrt{C}(R_L + R_C)}\right)} \\ &= \frac{Q^2 (R_L + R_C)}{1 + iQ\left(\omega_n - \frac{1}{\omega_n}\right)}, \end{aligned} \quad (3)$$

where the dimensionless  $Q = \frac{\sqrt{L}}{\sqrt{C}(R_L + R_C)}$  is called the quality factor of the circuit and  $\omega_n = \omega \sqrt{LC}$  is the normalized frequency.

Incidentally, it is noted that

$$\frac{1}{Q} = \frac{1}{Q_L} + \frac{1}{Q_C}, \quad (4)$$

where  $Q_L = \frac{\sqrt{L}}{\sqrt{C}R_L}$  is the quality factor of the coil and  $Q_C = \frac{\sqrt{L}}{\sqrt{C}R_C}$  is the quality factor of the capacitor. Traditionally, the inductor is considered low loss if  $Q_L \geq 10$ , whereas the same is true for the capacitor for  $Q_C \geq 10$ . Please recall that the analysis in this paper requires both components to be low loss.

Finally in this section, using Eq. (3), it will be useful to define the normalized impedance as

$$\frac{Z_p}{R_L + R_C} = \frac{Q^2}{1 + iQ\left(\omega_n - \frac{1}{\omega_n}\right)}. \quad (5)$$

### A. Magnitude of the Impedance

Using Eq. (5), the magnitude of the normalized impedance becomes

$$\frac{|Z_p|}{R_L + R_C} = \frac{Q^2}{\sqrt{1 + Q^2\left(\omega_n - \frac{1}{\omega_n}\right)^2}}. \quad (6)$$

Clearly, Eq. (6) is maximized when the denominator is minimized which occurs when  $\omega_{nmax} = 1$ , i.e.,

$$\omega_{max} = \frac{1}{\sqrt{LC}}. \quad (7)$$

Hence, from Eq. (6), it follows that the maximum impedance magnitude is given by

$$|Z_p|_{max} = Q^2 (R_L + R_C). \quad (8)$$

### B. Bandwidth of the Impedance Magnitude Response

A plot of the magnitude of the normalized impedance as a function of normalized frequency for various  $Q$  values is given in Figure 2.

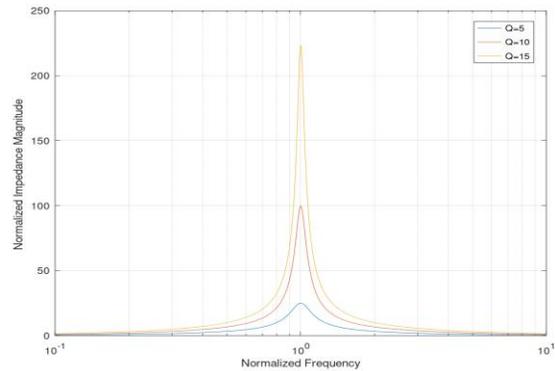


FIGURE 2. Plots of normalized impedance magnitude as a function of normalized frequency for various values of  $Q$ .

From this figure, it is clear that the narrowness of the peak of the impedance magnitude response depends upon the value of  $Q$ . Conventionally, the way to measure this is the bandwidth (BW) which is defined as

$$BW = \omega_2 - \omega_1, \quad (9)$$

where  $\omega_k$  ( $k = 1, 2$ ) are the two cut-off frequencies for which the normalized impedance magnitude is  $1/\sqrt{2}$  its maximum value. Note that  $\omega_2 > \omega_1$ .

Hence, from Eq. (6),

$$\frac{|Z_p|}{R_L + R_C} = \frac{Q^2}{\sqrt{1 + Q^2\left(\omega_{nk} - \frac{1}{\omega_{nk}}\right)^2}} = \frac{Q^2}{\sqrt{2}}. \quad (10)$$

Taking the reciprocal of Eq. (10) and squaring gives

$$\left(\omega_{nk} - \frac{1}{\omega_{nk}}\right)^2 = \frac{1}{Q^2}. \quad (11)$$

Hence,

$$\left(\omega_{nk} - \frac{1}{\omega_{nk}}\right) = \frac{1}{Q}. \quad (12)$$

or

$$\left(\omega_{nk} - \frac{1}{\omega_{nk}}\right) = -\frac{1}{Q}. \quad (13)$$

$$\omega_o \approx \frac{1}{\sqrt{LC}} = \omega_{max}. \quad (23)$$

Rewriting Eq. (12) as a quadratic gives

$$\omega_{nk}^2 - \frac{1}{Q}\omega_{nk} - 1 = 0. \quad (14)$$

Solving this quadratic for  $k = 2$  results in

$$\omega_{n2} = \frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}. \quad (15)$$

or

$$\omega_2 = \frac{1}{\sqrt{LC}} \left( \frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}} \right). \quad (16)$$

Rewriting Eq. (13) as a quadratic gives

$$\omega_{nk}^2 + \frac{1}{Q}\omega_{nk} - 1 = 0. \quad (17)$$

Solving this quadratic for  $k = 1$  results in

$$\omega_{n1} = -\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}. \quad (18)$$

or

$$\omega_1 = \frac{1}{\sqrt{LC}} \left( -\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}} \right). \quad (19)$$

Using Eq. (9), Eq. (16) and Eq. (19) gives the desired bandwidth as

$$BW(rad/s) = \frac{1}{\sqrt{LC}Q}, \quad (20)$$

or the bandwidth in Hz as

$$BW(Hz) = \frac{1}{2\pi\sqrt{LC}Q}. \quad (21)$$

### C. Frequency of Zero Phase of the Impedance

One way to define the resonant frequency is to state that it is the frequency at which the impedance phase is zero. The phase of the impedance of the circuit in Figure 1 can be shown [Section 16.4 of 3] to be zero for

$$\omega_o = \frac{1}{\sqrt{LC}} \sqrt{\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}}}. \quad (22)$$

Note that for a low loss inductor, as mentioned earlier,  $\sqrt{L/C}/R_L \geq 10$  and hence the numerator in the radical is approximately  $-L/C$ . Likewise, for a low loss capacitor,  $\sqrt{L/C}/R_C \geq 10$  and hence the denominator in the rightmost radical is approximately  $-L/C$ . Therefore, Eq. (22) becomes

Equation (23) confirms that the frequency of maximum impedance magnitude also occurs at the frequency for which the impedance phase is zero.

### III. VERIFICATION OF THE THEORETICAL EXPRESSIONS WITH PSPICE SIMULATION

To verify the theoretical expressions above, PSpice simulations will be performed in this section. (PSpice is electrical/electronic circuit simulation software that is well-known to electrical engineers and physicists. In this paper, PSpice ver. 9.1 is used. To download, please see [4], for example).

For the said simulations, the components of the circuit of Figure 1 are chosen to be  $C = 1\mu F, L = 25.33mH, R_C = 10\Omega$  and  $R_L = 15\Omega$ . These choices give  $Q_L = 10.61, Q_C = 15.92$  and  $Q = 6.367$ .

The simulated impedance magnitude is shown in Figure 3. Note that the cut-off points on the graph of Figure 3 are located by using the PSpice search command SFLE (721.037). The 721.037 comes from the simulated maximum impedance magnitude divided by  $\sqrt{2}$ .

The simulated impedance phase is shown in Figure 4, where the zero phase frequency is identified.

These simulated values are compared with their theoretical values in Table I.

TABLE I. Comparison of theoretical and PSpice simulated values.

Relevant Equation	Theoretical Value	Simulated Value
Eq. (7)	$f_{max} = \frac{\omega_{max}}{2\pi} = 1000$ Hz	1000 Hz
Eq. (19)	$f_1 = \frac{\omega_1}{2\pi} = 924.6$ Hz	924.5 Hz
Eq. (16)	$f_2 = \frac{\omega_2}{2\pi} = 1081.6$ Hz	1081.6 Hz
Eq. (23)	$f_o = \frac{\omega_o}{2\pi} = 1000$ Hz	997.5 Hz
Eq. (21)	$BW = 157.1$ Hz	1081.6-924.5=157.1 Hz
Eq.(8)	$ Zp _{max} = 1013 \Omega$	1020 $\Omega$

As can be seen from Table I, there is excellent agreement between the theoretical and simulated values.

### IV. CONCLUSIONS

The parallel resonant circuit with a low loss inductor and a low loss capacitor has been analyzed. The maximum impedance magnitude and the frequency at which it occurs have been determined without using calculus. The bandwidth of the parallel resonant circuit was also determined. Also, the frequency at which the impedance has zero angle has been found to be the frequency at which the impedance magnitude is maximum. PSpice simulations were performed which verified the theoretical findings.

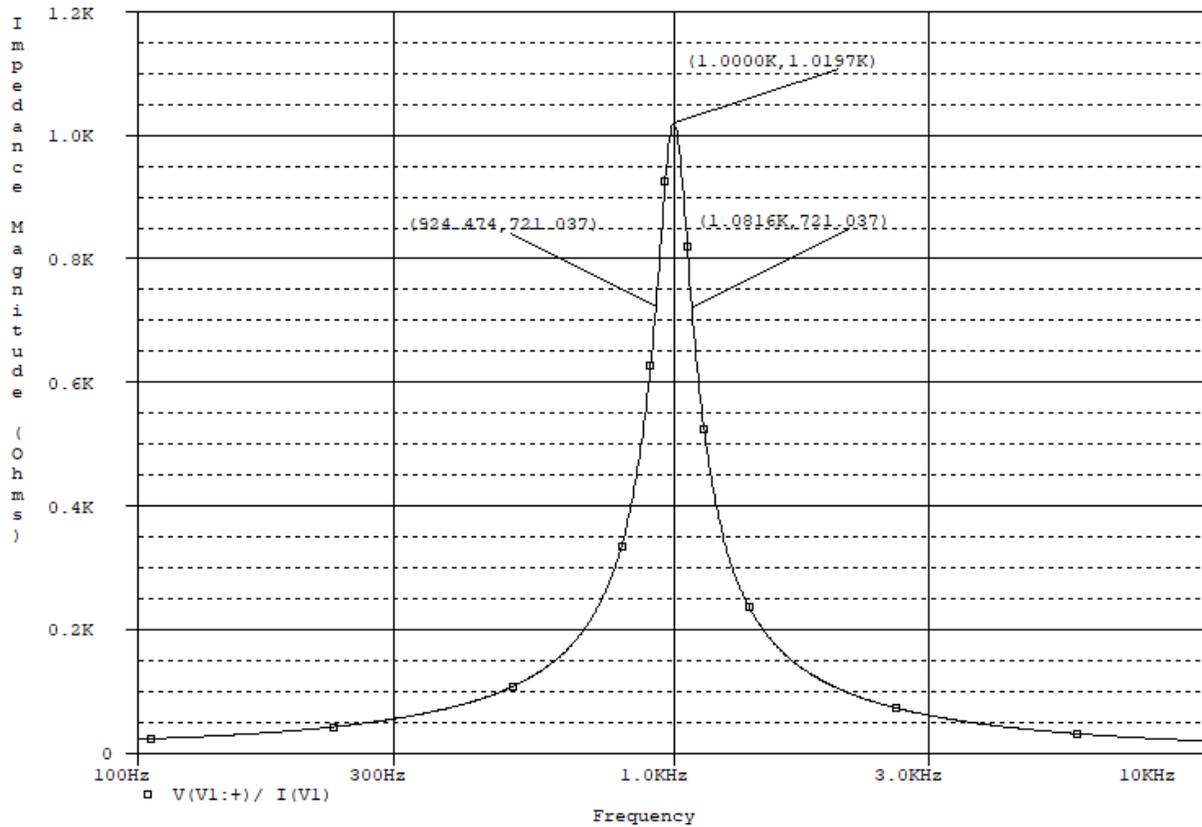


FIGURE 3. PSpice simulated magnitude of the input impedance (in  $\Omega$ ) for the parallel resonant circuit of Figure 1 with  $R_L = 15\Omega$ ,  $R_C = 10\Omega$ ,  $C = 1\mu F$  and  $L = 25.33mH$ . Please note that  $Q_L = 10.61$ ,  $Q_C = 15.92$  and  $Q = 6.367$ .

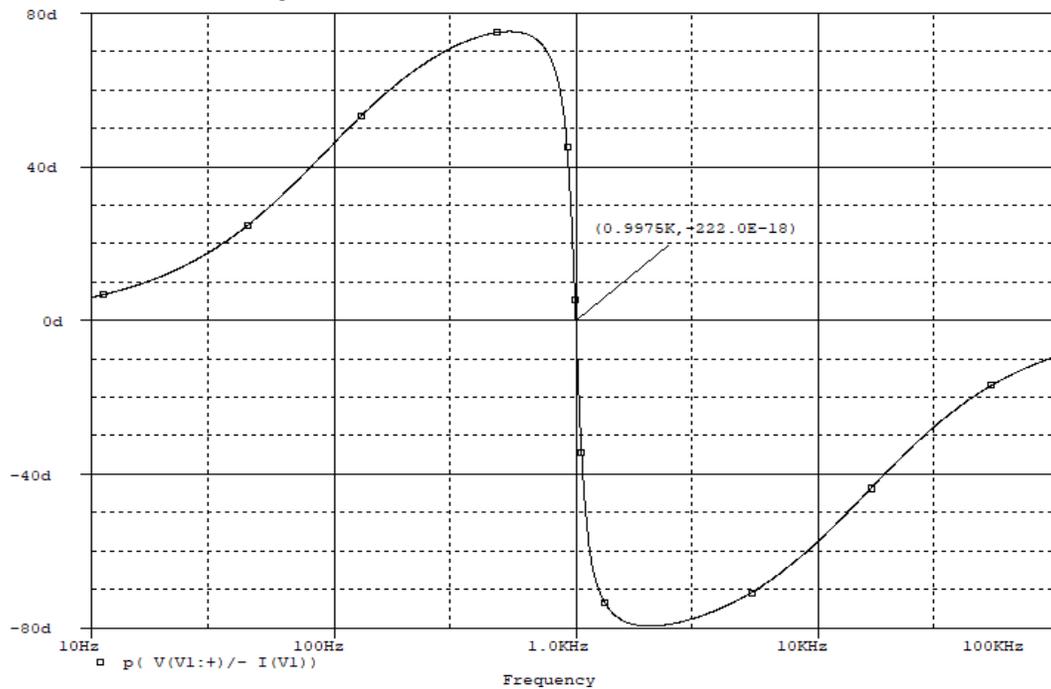


FIGURE 4. PSpice simulated phase (in deg) of the input impedance for the parallel resonant circuit of Figure 1 with the same components as in Figure 3.

## REFERENCES

- [1] Simon, A., “*Teaching RLC parallel circuit in high-school physics class,*” *Acta Didactica Napocensia* **8**, 33-46 (2015).
- [2] Cartwright, K. V., Joseph, E. & Kaminsky, E., “*Finding the Exact Maximum Impedance Resonant*

*Frequency of a Practical Parallel Resonant Circuit Without Calculus*”, *Technology Interface International Journal* **11**, 26-34 (2010). available in [Microsoft Word - TIJJVol11N1fall2010final4.DOC](#)

[3] [Document \(azureedge.net\)](#).

[4] *Installing PSpice 9.1-Dr. Montoya's Webpage*, [http://montoya.sdsmt.edu/ee220/handouts/PSpice91\\_install.pdf](http://montoya.sdsmt.edu/ee220/handouts/PSpice91_install.pdf).