Something else about what is not usually told in problems of inclined planes with friction

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QVO NON ASCENDAM?

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Abstract

Since Galileo's pioneering study of inclined planes, numerous research publications have focused on teaching this topic. This article is based on a previous publication in which we examined the work done by friction on an inclined plane and on a polygonal shape formed by two inclined planes joined at the vertex. Our current objective is to address certain aspects related to the velocity of an object sliding with friction on a horizontal plane and on an inclined plane. The analysis presented here is not typically found in university physics textbooks, nor is it included in exercise guides for basic physics courses in fields such as engineering or science. Based on the classical model of friction between a block and the surface of an inclined plane, this study leads to results that spark curiosity, some of which are unexpectedly counterintuitive.

Keywords: Inclined plane, Horizontal plane, Friction force, Velocity.

Resumen

Desde el estudio pionero de Galileo sobre los planos inclinados, numerosas publicaciones de investigación se han centrado en la enseñanza de este tema. Este artículo se basa en una publicación previa en la que examinamos el trabajo realizado por la fricción en un plano inclinado y en una figura poligonal formada por dos planos inclinados unidos en el vértice. Nuestro objetivo actual es abordar ciertos aspectos relacionados con la velocidad de un objeto que se desliza con fricción en un plano horizontal y en un plano inclinado. El análisis que aquí se presenta no suele encontrarse en los libros de texto universitarios de física, ni se incluye en las guías de ejercicios para cursos básicos de física en campos como la ingeniería o las ciencias. Basado en el modelo clásico de fricción entre un bloque y la superficie de un plano inclinado, este estudio arroja resultados que despiertan curiosidad, algunos de los cuales son inesperadamente contraintuitivos.

Palabras clave: Plano inclinado, Plano horizontal, Fuerza de fricción, Velocidad.

I. INTRODUCTION

Since Galileo's study on inclined planes, countless research publications have focused on teaching this topic (1-10). In a previous publication (10), we analyzed the work done by friction on an inclined plane and on a polygonal shape formed by two inclined planes joined at the vertex. Our aim here is to explore certain aspects related to the velocity of a moving object sliding along a horizontal plane and an inclined plane, taking friction into account in both cases. The analysis presented here is usually not found in university textbooks or in exercise guides for introductory physics courses such as engineering or science degrees. This study focuses on how the classical model of friction between a block and an inclined plane can lead to surprising and sometimes counterintuitive results. We examine how the velocity of a block changes when it slides along a horizontal path and an inclined plane. In both cases, we start with a particular initial velocity and assume friction between the block and the surface. We also analyze the time taken when the block travels along the horizontal surface or the inclined plane. In the case of the inclined plane, we study the block's velocity as the angle varies, while keeping the height, base, or length of the inclined plane constant. Each of these cases has practical applications such as ramp design, material transport in factories, or downhill sports. These applications can benefit from this detailed analysis of how angle and friction influence motion. Furthermore, these insights can help connect theoretical findings to problems in engineering and applied physics.

II. DISCUSSION

Let's consider a block of mass M sliding down an inclined plane with friction and length L, forming an angle α with the horizontal plane. The forces acting on the block are weight P, constant dynamic friction Fr, and the normal force N.

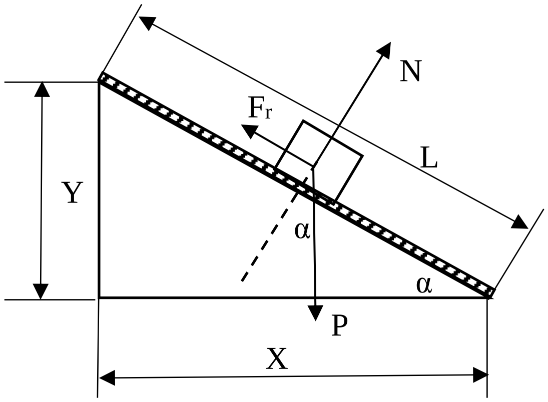

Figure 1: Block sliding on an inclined plane with friction

The magnitudes of these forces are given by $P = Mg$, $N = Mg\cos(\alpha)$ and $Fr = \mu_D Mg\cos(\alpha)$.

In this work we analyze two cases in depth. In the first case we study the block's velocity on a horizontal plane because there is a particular initial velocity that yields a surprising conclusion when used on the inclined plane. In the second case, we study the velocity of the block when we consider α≠0°, using the specific initial velocity found in the first case. Here, we vary α, keeping the height Y_T, the base X_T, or the length L of the inclined plane constant.

CASE 1: If α=0°, the initial velocity required for the block of mass M to travel the distance X_T and come to a stop is derived from an energy approach. If we impose that the final velocity is zero when the block travels distance X_T, then the initial velocity must be equal to:

$$V_I = \sqrt{2\,\mu_D\,g\,X_T} \quad (1)$$

The acceleration for this case is $-\mu_D\,g$. Therefore, velocity as a function of position is expressed as:

$$V = \sqrt{V_I^2 + 2\,a\,X} = \sqrt{V_I^2 - 2\,\mu_D\,g\,X} \quad (2)$$

When the block is halfway the distance X_T, the velocity is:

$$V = \sqrt{V_I^2 - 2\,\mu_D\,g\,\frac{X_T}{2}} = \sqrt{V_I^2 - \frac{V_I^2}{2}} = \sqrt{\frac{V_I^2}{2}} = \sqrt{\frac{2\,\mu_D\,g\,X_T}{2}} = \sqrt{\mu_D\,g\,X_T} \quad (3)$$

Figure 2 shows the velocities as a function of the distance traveled in the horizontal plane. The green line corresponds to $\mu_D = 0.2$ and initial velocity 2 m/s. The yellow line corresponds to $\mu_D = 0.8$ and 4 m/s.

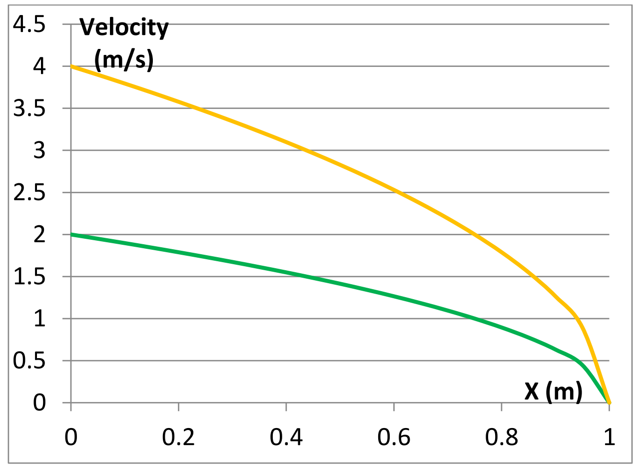

Figure 2: Evolution of velocities on the horizontal plane in terms of distance traveled for initial velocity of 4m/s with $\mu_D = 0.8$ (yellow curve) and 2m/s with $\mu_D = 0.2$ (green curve)

Solving for position in Equation (2), we can find where the block is located when its velocity is halved from the initial velocity:

$$X = \frac{\left(\frac{V_I}{2}\right)^2 - V_I^2}{-2\,\mu_D\,g} = \frac{V_I^2\left(\frac{1}{4}-1\right)}{-2\,\mu_D\,g} = \frac{2\,\mu_D\,g\,X_T\left(-\frac{3}{4}\right)}{-2\,\mu_D\,g} = \frac{3}{4}X_T \quad (4)$$

Consequently, whenever the initial velocity is as given by Equation (1), the block's velocity is halved after it travels 75% of the total distance. Also, when the block reaches the midpoint, its velocity depends on the friction coefficient and is equal to $\sqrt{\mu_D\,g\,X_T}$.

Using the previously calculated acceleration $(-\mu_D\,g)$, velocity as a function of position (Equation (2)), and velocity as a function of time $(V = V_I - \mu_D\,g\,t)$, we can derive time as a function of position:

$$t = \frac{V_I - V}{\mu_D\,g} = \frac{\sqrt{2\,\mu_D\,g\,X_T} - \sqrt{V_I^2 - 2\,\mu_D\,g\,X}}{\mu_D\,g} = \frac{\sqrt{2\,\mu_D\,g\,X_T} - \sqrt{2\,\mu_D\,g\,X_T - 2\,\mu_D\,g\,X}}{\mu_D\,g} \quad (5)$$

When the position is $\frac{3}{4}X_T$ (which corresponds to half the initial velocity), the time is:

$$t = \frac{\sqrt{2\,\mu_D\,g\,X_T} - \sqrt{2\,\mu_D\,g\,X_T - 2\,\mu_D\,g\,\frac{3}{4}X_T}}{\mu_D\,g} = \sqrt{\frac{X_T}{\mu_D\,g}}\,\frac{1}{\sqrt{2}} \qquad (6)$$

The time it takes for the block to travel distance X_T = 1m and stop is shown in Figure 3. This graph displays time for two values of the dynamic friction coefficient ($\mu_D = 0.2$ and $\mu_D = 0.8$):

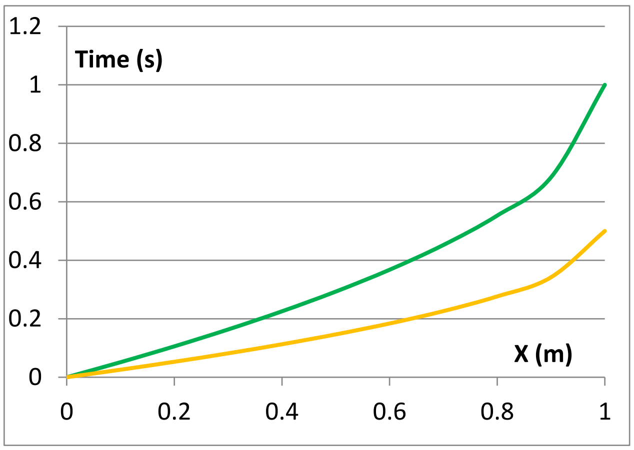

Figure 3: Comparison of time to travel distance 0m<X<1m on a horizontal plane. Green curve: $\mu_D = 0.2$ and initial velocity 2m/s.Yellow curve: $\mu_D = 0.8$ and initial velocity 4m/s

The green line represents $\mu_D = 0.2$ and initial velocity 2m/s. The yellow line represents $\mu_D = 0.8$ and 4m/s. The graph shows that the block takes 1s to travel 1m when $\mu_D = 0.2$ and initial velocity 2m/s and 0.5s when $\mu_D = 0.8$ and initial velocity 4m/s. Additionally, when the time is half the total travel time, the block's position is 0.75m in both cases. Based on these cases, we conclude that whenever the block is launched with the initial velocity given by Equation (1) it covers 75% of the total distance in half the total time.

CASE 2: If $\alpha \neq 0°$, we consider three cases: constant X_T, constant Y_T, and constant L. The block's acceleration is determined by the net force in the inclined plane's direction, so the acceleration is:

$$a = g\,\text{sen}(\alpha) - \mu_D\,g\cos(\alpha) \qquad (7)$$

The block's final velocity after traveling the length of the plane L is:

$$V_F = \sqrt{V_I^2 + 2\left(g\,\text{sen}(\alpha) - \mu_D\,g\cos(\alpha)\right)L} \qquad (8)$$

Considering the case of constant X_T, the final velocity is:

$$V_F = \sqrt{V_I^2 + 2\,g\,X_T\,\text{tg}(\alpha) - 2\,\mu_D\,g\,X_T} \qquad (9)$$

If we use the initial velocity found in CASE 1, which has the value $\sqrt{2\,\mu_D\,g\,X_T}$, the final velocity becomes:

$$V_F = \sqrt{2\,g\,X_T\,\text{tg}(\alpha)} \qquad (10)$$

At this point, we encounter a curious observation: for the initial velocity given by Equation (1), the final velocity is independent of the friction coefficient and depends only on the inclined plane's angle. For any other initial velocity, the final velocity would depend on the friction coefficient.

The following Graph 3 shows the final velocity of the block as a function of the inclined plane's angle, with constant X_T set to 1m, g= 10m/s^2 and μ_D= 0.2.

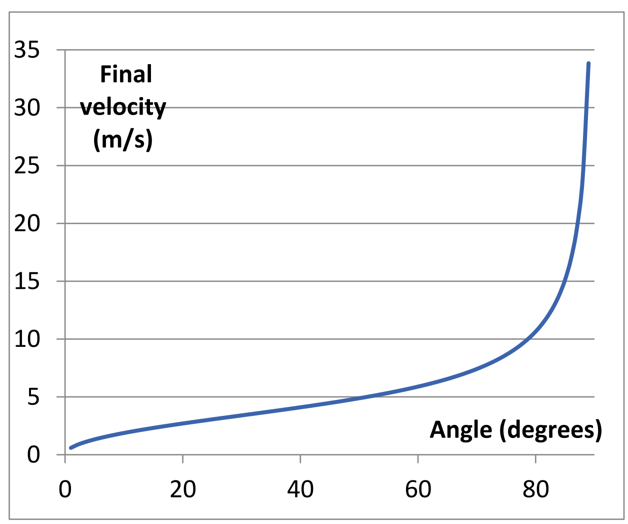

Figure 4: Final velocity of the block as a function of the inclined plane's angle when the initial velocity is $\sqrt{2\,\mu_D\,g\,X_T}$

In Figure 5, we observe how the block's velocity varies for angles of 30°, 45° and 60° as it slides down the inclined plane. When the block reaches the base (with X_T=1m), we see that the final velocity is non-zero and remains the same for each angle of inclination despite different friction coefficients and initial velocities.

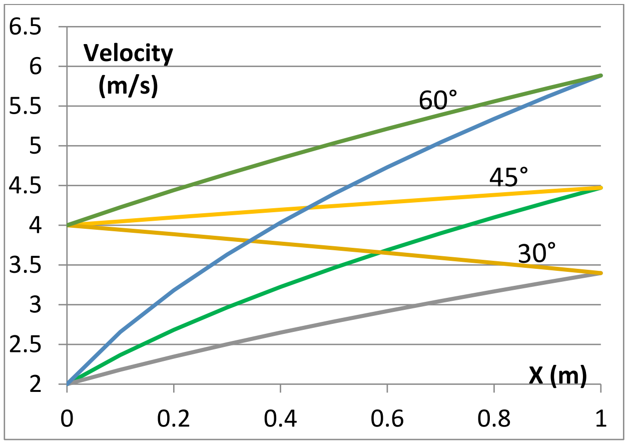

Figure 5: Evolution of velocities as a function of distance traveled, with an initial velocity of 2m/s and μ_D= 0.2 for angles of 30°, 45° and 60°; and an initial velocity of 4m/s and μ_D= 0.8 for angles of 30°, 45° and 60°

The time it takes for the block to travel length L is given by:

$$t = \frac{V_F - V_I}{a} = \frac{\sqrt{V_I^2 + 2\, g\, X_T\, tg(\alpha) - 2\, \mu_D\, g\, X_T} - V_I}{g\, sen(\alpha) - \mu_D\, g\, \cos(\alpha)} \qquad (11)$$

Figure 6 shows the time required for the block to travel distance L as the inclined plane angle varies when X_T is constant. The initial velocity is given by Equation (1), and its value is 2m/s when X_T=1m, $\mu_D = 0.2$ and g=10 m/s^2.

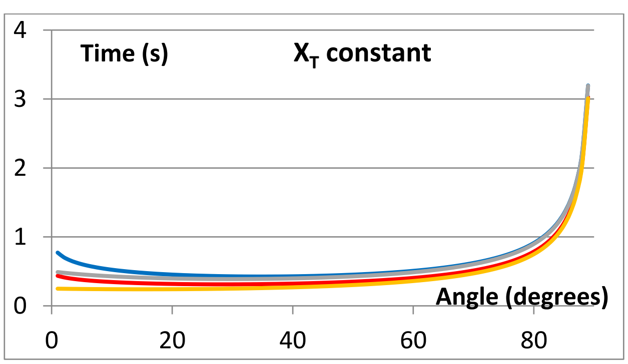

Figure 6: Time for the block to travel length L on the inclined plane as a function of the angle, for an initial velocity given by Equation (1). Blue line with μ_D = 0.2; red line with μ_D = 0.8. Green and yellow lines correspond to cases without friction and initial velocities of 2m/s and 4m/s, respectively.

The blue line in Figure 6 corresponds to $\mu_D = 0.2$ and the red line to $\mu_D = 0.8$. As the plane's angle increases, the time to travel distance L decreases to a minimum and then increases, approaching infinity. Additionally, as the friction coefficient increases, the time to cover the same distance L is shorter for higher friction coefficients, with the difference more pronounced at smaller angles. Figure 6 also shows times without friction. The green line corresponds to an initial velocity of 2m/s, and the yellow line to 4m/s. The difference in travel time for the same initial velocity, with and without friction, is greater for smaller angles (blue and green curves for 2m/s, red and yellow for 4m/s).

In conclusion, we observe that, for different values of the friction coefficient, the block reaches the same final velocity but takes different times to reach the base of the plane. Additionally, the minimum time to travel the plane occurs at angles below 45°, and the displacement is greater for higher friction coefficients. Figure 4 also shows that there is an angle at which the velocity remains constant. If the velocity is constant the acceleration is zero and by equating Equation (1) with Equation (10) we find that μ_D = tg(α). For a friction coefficient of μ_D = 0.8 the angle α is 38.7° and for μ_D = 0.2 the angle at which the velocity remains constant is 11.3°. For each friction coefficient there is an angle at which the velocity is constant. Figure 7 shows the case when μ_D = 0.8.

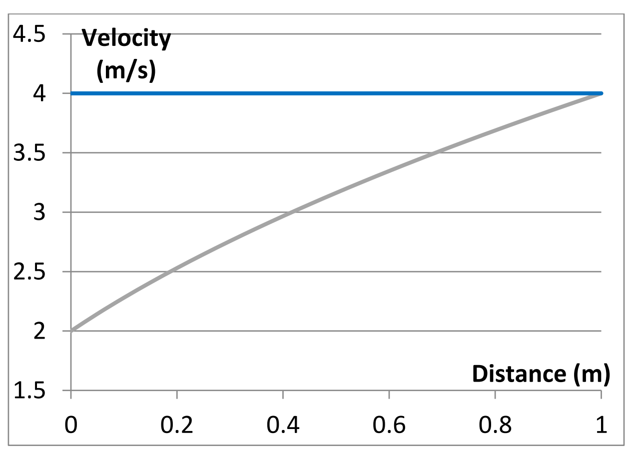

Figure 7: Velocity evolution as a function of distance. Blue line: initial velocity of 4m/s, friction coefficient μ_D= 0.8 and α= 38.7°. Green line: initial velocity of 2m/s, friction coefficient μ_D= 0.2 and α=38.7°

If we consider the case with constant height Y_T, the final velocity is:

$$V_F = \sqrt{V_I^2 + 2\left(g\,Y_T - \mu_D\,g\,\frac{Y_T}{tg(\alpha)}\right)} \tag{12}$$

If the initial velocity is as in Equation (1), the final velocity becomes:

$$V_F = \sqrt{2\,\mu_D\,g\,\frac{Y_T}{tg(\alpha)} + 2\,g\,Y_T - 2\,\mu_D\,g\,\frac{Y_T}{tg(\alpha)}} = \sqrt{2\,g\,Y_T} \tag{13}$$

This leads to another curious result: the final velocity is independent of both the angle and the friction coefficient; it depends only on height Y_T. Figure 8 shows the velocities (as the block descends the plane) for three different angles, with Y_T= 1m. We can see that the velocities reach different values but converge to the same final velocity at the plane's base.

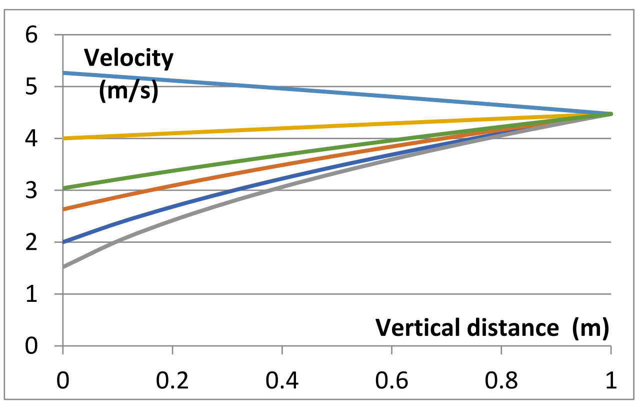

Figure 8: Block velocity evolution as a function of vertical distance. The lower three curves correspond to μ_D = 0.2, with angles 60°, 45° and 30° in ascending order. The upper three curves correspond to μ_D = 0.8, with angles 60°, 45° and 30° in ascending

The time taken by the block to cover length L is:

$$t = \frac{V_F - V_I}{a} = \frac{\sqrt{2\,g\,Y_T} - V_I}{g\,sen(\alpha) - \mu_D\,g\cos(\alpha)} \tag{14}$$

If the initial velocity is given by Equation (1), we obtain:

$$t = \frac{\sqrt{2\,g\,Y_T} - \sqrt{2\,\mu_D\,g\,\frac{Y_T}{tg(\alpha)}}}{g\,sen(\alpha) - \mu_D\,g\cos(\alpha)} \tag{15}$$

In Figure 9, the time required to travel distance L is shown as the plane's angle varies, with Y_T=1m. The initial velocity is given by Equation (1), for this case is given by $\sqrt{2\,\mu_D\,g\,\frac{Y_T}{tg(\alpha)}}$.

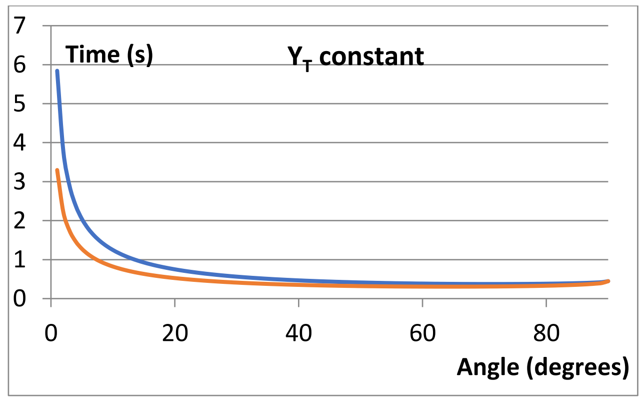

Figure 9: Travel time as a function of plane angle for constant Y_T with Y_T=1m. Blue line corresponds to μ_D = 0.2; red line corresponds to μ_D = 0.8

In conclusion for this case, we have that the final velocity depends only on height and matches the velocity achieved in free fall when the initial velocity on the plane is given by Equation (1). The travel time is always shorter with a higher friction coefficient, reaching a minimum between angles of 65° and 75°, with the higher coefficient occurring at a lower angle.

On the other hand, if L is kept constant, the final velocity is:

$$V_F = \sqrt{V_I^2 + 2\left(g\,sen(\alpha) - \mu_D\,g\cos(\alpha)\right)L} \tag{16}$$

If we use the initial velocity given in Equation (1), the final velocity becomes:

$$V_F = \sqrt{2\,\mu_D\,g\,L\cos(\alpha) + 2\left(g\,sen(\alpha) - \mu_D\,g\cos(\alpha)\right)L} = \sqrt{2\,g\,L\,sen(\alpha)} \tag{17}$$

From this, we conclude that the final velocity does not depend on the friction coefficient but does depend on the angle and length L of the plane.

Figure 10 shows the velocities for three different angles, assuming L=1m.

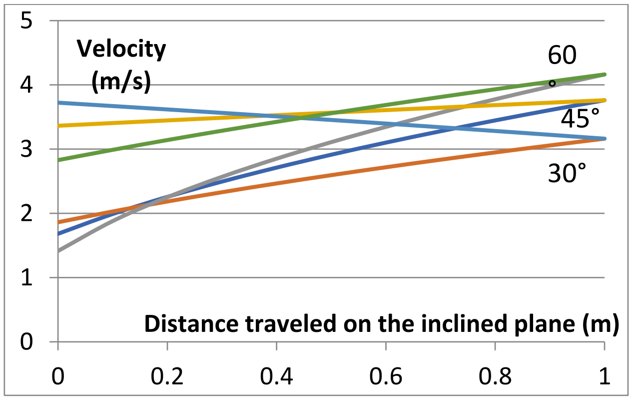

Figure 10: Velocity evolution as a function of distance traveled on the inclined plane. The three lower lines correspond to $\mu_D = 0.2$ with angles of 60°, 45° and 30° in ascending order. The three upper lines correspond to $\mu_D = 0.8$ with angles of 60°, 45° and 30° in ascending order.

In this graph, we observe that, for any angle on the inclined plane, the final velocity is the same for all friction coefficients. Additionally, the final velocity increases as the angle of inclination increases.

The time taken by the block to cover length L, when it is kept constant, is given by:

$$t = \frac{V_F - V_I}{a} = \frac{\sqrt{2\,g\,L\,sen(\alpha)} - V_I}{g\,sen(\alpha) - \mu_D\,g\cos(\alpha)} \tag{18}$$

If the initial velocity is given by Equation (1):

$$t = \frac{\sqrt{2\,g\,L\,sen(\alpha)} - \sqrt{2\,\mu_D\,g\,L\cos(\alpha)}}{g\,sen(\alpha) - \mu_D\,g\cos(\alpha)} \tag{19}$$

In Figure 11 the time it takes for the block to travel distance L is shown as the angle of the inclined plane varies assuming L=1m. The initial velocity is given by Equation [1] that in this case is $\sqrt{2\,\mu_D\,g\,L\cos(\alpha)}$.

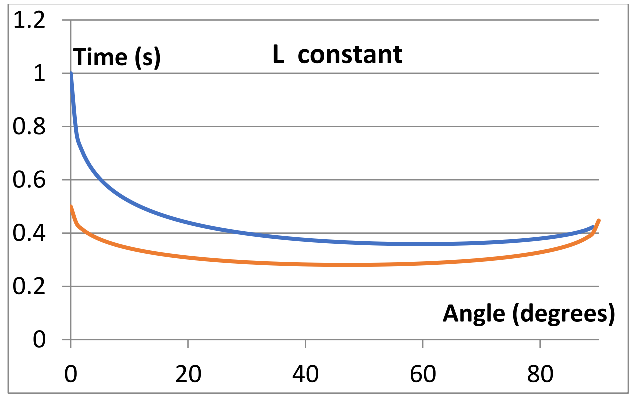

Figure 11: Travel time as a function of the plane angle for constant L with L=1m. The blue line corresponds to $\mu_D = 0.2$; the red line corresponds to $\mu_D = 0.8$

In conclusion for this case the final velocity depends on the length L and the angle of inclination but not on the friction coefficient. Here, as in the previous case, the travel time is always shorter for higher friction coefficients. Additionally, the minimum time occurs at angles greater than 45°, with the higher coefficient at a lower angle. We also reach another curious result: for these conditions in the time vs. angle graphs, the block's travel time is shorter with a friction coefficient of 0.8 than with a coefficient of 0.2.

Finally, Figure 12 shows the final velocities as a function of the angle for constant X_T, Y_T and L assuming $X_T = Y_T = L = 1$m.

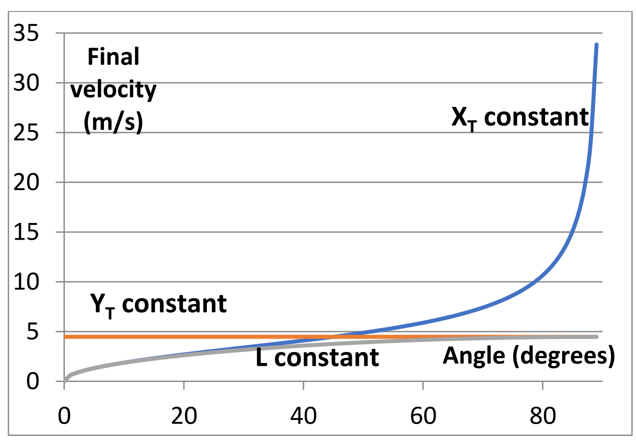

Figure 12: Final velocity values of the block as a function of the angle on the inclined plane when X_T, Y_T and L are all constant and equal to 1 m

From Figure 12, we observe that at an angle of 45°, the final velocity for constant X_T is equal to the final velocity for constant Y_T. For small angles, the velocity for constant X_T matches that for constant L. For large angles, the velocity for constant Y_T matches that for constant L. Additionally, the final velocity for constant L is always lower than the final velocity for constant Y_T.

III. CONCLUSIONS

This study addresses topics that are typically not covered in exercise guides for introductory university physics courses for engineering or science majors when dealing with problems involving blocks sliding down inclined planes with friction. These analyses are also usually absent in basic university and pre-university physics textbooks. The analysis in this study requires only basic knowledge of kinematics, dynamics and energy. Additionally, this work follows the analysis previously conducted and referenced as [10]. This study is based on the classical model of friction between a block and an inclined plane, leading to results that are striking, even counterintuitive. For example, the time intervals obtained for a friction coefficient of 0.8 are shorter than those for a friction coefficient of 0.2. Another intriguing result is that, under certain conditions, the final velocity of the block on the inclined plane does not depend on the friction coefficient but only on the angle of inclination. Finally, another surprising result is observed when comparing the travel times across the three cases studied. In all cases, the block's travel time is shorter with a friction coefficient of 0.8 than with a friction coefficient of 0.2.

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